Exercises to the course:

Introduction to nonlinear geometric PDEs

- Part I -

Exercise 1.1 (**Geometric meaning of the principal curvatures**). *Let M be a smooth surface in $\mathbb{R}^3$ with unit normal ν. Let $\varepsilon > 0$ and $\gamma : (-\varepsilon, \varepsilon) \to M$ be a smooth curve such that $\gamma(0) = p$ and $\gamma'(0) = v$ with $|v| = 1$. We define the curvature of M at p in direction v as*

$$k_p : T_pM \to \mathbb{R} : v \mapsto k_p(v) := \langle \gamma''(0), \nu \rangle.$$

Note that by Meusnier's theorem k_p is well defined. Answer the following questions

(i) Why does it make sense to call k_p the curvature of M at p in direction v?

(ii) What is the relation between k_p and h?

(iii) What are the critical values of k_p in terms of h and g?

Exercise 1.2 (**Graphical hypersurfaces in $\mathbb{R}^n$**). *Suppose that M is a hypersurface in $\mathbb{R}^{n+1}$ which is given as a graph over a chart domain $U \subset \mathbb{R}^n$: $M = \operatorname{graph} u$. Verify the following formulae:*

$$g_{ij} = \delta_{ij} + D_i u D_j u, \qquad g^{ij} = \delta^{ij} - \frac{D^i u D^j u}{1 + |Du|^2}, \qquad \Gamma^k_{ij} = \frac{D_{ij} u D^k u}{1 + |Du|^2},$$

$$\nu = \frac{\pm 1}{\sqrt{1 + |Du|^2}} \left(-Du, 1\right), \qquad h_{ij} = \frac{\pm D_{ij} u}{\sqrt{1 + |Du|^2}},$$

$$H := g^{ij} h_{ij} = \frac{\pm 1}{\sqrt{1 + |Du|^2}} \left(\delta^{ij} - \frac{D^i u D^j u}{1 + |Du|^2} \right) D_{ij} u = \operatorname{div}_{\mathbb{R}^{n+1}} \left(\frac{\pm Du}{\sqrt{1 + |Du|^2}} \right).$$

Exercise 1.3 (**Explicit computation**). *Compute the principal curvatures, the mean curvature and the Gaussian curvature of the graph of the function $u : B_2(0) \to \mathbb{R} : (x, y) \mapsto x^2 - y^2$ at the origin.*

Exercise 1.4 (**Counter examples**). *(i) Let $R > 0$ and $\Omega = B_R(0) \subset \mathbb{R}^n$. Is there always a function $u : \Omega \to \mathbb{R}$ with $u = 0$ on $\partial\Omega$ such that the graph of u has constant mean curvature $H = c$?*

(ii) Let $0 < R_1 < R_2 < \infty$ and $\Omega = B_{R_2}(0) \setminus \overline{B_{R_1}(0)}$. Is there always a function $u : \overline{\Omega} \to \mathbb{R}$ with $u = 0$ on $\{|x| = R_2\}$ and $u = L > 0$ on $\{|x| = R_1\}$ such that the graph of u has zero mean curvature? Hint: Try to find an explicit solution.

Exercise 1.5 (**Flat v.s. harmonic v.s. minimal**). *Let $\Omega = B_2(0) \setminus \overline{B_1(0)}$ and the boundary conditions be $u = 0$ on $\{|x| = 2\}$ and $u = 1$ on $\{|x| = 1\}$. Compare the surface area of the graphical minimal surface* $\operatorname{graph} v_m$ *to that of the truncated cone, i.e. to* $\operatorname{graph} v_c$ *where $v_c(x) := 2 - |x|$ as well as to* $\operatorname{graph} v_h$ *where v_h is the corresponding harmonic function, i.e. the function satisfying $\Delta v_h = 0$ in Ω together with the same boundary values.*

Exercise 1.6 (**Derivative of the determinant**). *Let* $\varepsilon > 0$. *Suppose that*

$$B \in C^1\Big((t_0 - \varepsilon, t_0 + \varepsilon), \mathbb{R}^{n\times n}\Big)$$

and that $B(t_0)$ *is invertible. Show that*

$$\frac{d}{dt}\Big|_{t=t_0} \det B(t) = \det B(t_0) \operatorname{tr}\left(B^{-1}(t_0) \frac{d}{dt}\Big|_{t=t_0} B(t)\right)$$

holds.

Exercise 1.7 (**Understanding Hölder spaces**).

(i) Can you find a function which is in $C^{0,\alpha}(\overline{\Omega})$ *but not in* $C^{0,\alpha+\varepsilon}(\overline{\Omega})$ *for* $\varepsilon > 0$*? Can you find a function that is in* $C^{1,1}(\overline{\Omega})$ *but not in* $C^2(\overline{\Omega})$*?*

(ii) Can you find a domain $\Omega \subset \mathbb{R}^2$ *and a function* $u \in C^1(\overline{\Omega})$ *which is not in* $C^{0,3/4}(\overline{\Omega})$*?*

(iii) Why do we need to work with Hölder spaces $C^{k,\alpha}(\overline{\Omega})$ *instead of the easier* $C^k(\overline{\Omega})$ *spaces?*