Exercises to the lecture:

Introduction to nonlinear geometric PDEs

- Part II -

Exercise 2.1.★ *Try to prove Corolloary 4.3, i.e.:*

> *Let $(X, \|\cdot\|)$ be a Banach space and $A \subset X$ closed and convex. If $T : A \to A$ is continuous and TA is relativly compact then T has a fixed point.*

Hint: Try to apply Schauder's theorem. Is the set $(\overline{TA})^{co}$ compact?

Exercise 2.2 (**C^0-estimate**). *Use Theorem 5.9 to prove a C^0-a priori estimate for solutions of the prescribed mean curvature problem* (PMC). *Which condition do you have to impose on the right hand side, i.e. on H?*

Exercise 2.3. *Let $v(Du) := \sqrt{1+|Du|^2}$ and $a^{ij}(Du) := \partial a^i/\partial p^j|_{Du}$ where $a^i(p) := p^i/v(p)$. Let $w := |Du|^2/2$ and H the mean curvature of* graph u. *Verify the following formulae:*

$$a^{ij}D_iuD_ju = v^{-3}|Du|^2, \qquad a^{ij}D_iuD_jv = v^{-4}D^kwD_ku$$

$$a^{ij}D_{ik}uD_j^ku \geq 0, \qquad D_i(a^{ij}D_ju) = v^{-2}\Big(H - 2v^{-3}D^kuD_kw\Big).$$

Exercise 2.4 (**Interior gradient estimate:** $H = H(x, z, \nu)$)**.★** *Prove the interior gradient estimate, Theorem 5.14. Hint: Use the ansatz $\hat{w} := \log v + f(u)$ and compute $L\hat{w} := \Delta_M \hat{w}$ where Δ_M is the Laplace Beltrami operator on $M = \operatorname{graph} u$.*

Exercise 2.5 (**Distance function**). *Try to prove Lemma 5.20 on the Hessian of the distance function. In particular, verify that*

$$\Delta \operatorname{dist}(x, \partial\Omega) \leq -H_{\partial\Omega}(y)$$

where $y \in \partial\Omega$ is such that $\operatorname{dist}(x, \partial\Omega) = |x-y|$.

Exercise 2.6 (**Boundary gradient estimate**)**.★** *Try to prove Theorem 5.24, i.e. a boundary gradient estimate in the case where $H = H(x, z, \nu)$. Hint: Try a proof along the lines of the proof of Theorem 5.23. Show first that*

$$\left| H\Big(x, \phi(x), \nu(Dw^{\pm}(x))\Big) - H\left(y, \phi(y), \begin{pmatrix} \pm\mu(y) \\ 0 \end{pmatrix}\right)\right| \leq c_1 \operatorname{d}(x) + \frac{c_2}{\psi'(\operatorname{d}(x))}.$$

where $y \in \partial\Omega$ is such that $\operatorname{d}(x) = |x-y|$.

Exercise 2.7 (**Comparison principle**). *Try to prove the comparsion principle for fully nonlinear operators, Theorem 6.6, along the lines of the proof for the corresponding Theorem for quasilinear operators, Theorem 5.1.*

Exercise 2.8 (**Uniqueness**). *Try to prove that if in addition to the assumptions in Theorem 6.13 we require the a^{ij} to be independent of z then there exists at most one solution $u \in C^{2,\alpha}(\overline{\Omega})$ of the quasilinear oblique derivative problem. What can you say about the uniqueness for the fully nonlinear oblique derivative problem?*

Exercise 2.9. *Let* $M = \operatorname{graph} u$ *where* $u : \overline{\Omega} \to R$*. Let* $f, g \in C^1(\overline{\Omega})$*,* $W(Du) := \sqrt{1 + |Du|^2}$ *and* $a^{ij}(Du) := \partial a^i / \partial p^j|_{Du}$ *where* $a^i(p) := p^i / W(p)$*. Show that the following relations hold*

$$W a^{ij}(Du) D_i f D_j f = |\nabla f|^2, \qquad a^{ij}(Du) D_i f D_j g \leq W^{-1}(Du) |\nabla f| |Dg|$$

where $\nabla f := Df - \langle Df, \nu \rangle \nu$*.*