Exercises to the lecture:

Introduction to nonlinear geometric PDEs

- Part III -

Exercise 3.1 (**Short-time existence of MCF with Neumann condition**). *Show that*

$$\begin{cases} \dfrac{\partial u}{\partial t} = \sqrt{1+|Du|^2}\,\operatorname{div}\left(\dfrac{Du}{\sqrt{1+|Du|^2}}\right) & \text{in } \Omega\times(0,T) \\ D_\mu u = 0 & \text{on } \partial\Omega\times(0,T) \\ u = u_0 & \text{on } \Omega\times\{0\} \end{cases}$$

has a solution at least for a short time.

Exercise 3.2 (**Graphs over $\mathbb{S}^n$**). *Try to fill in the gaps in the proof of Lemma 8.19. about the geometry of graphs over $\mathbb{S}^n$.*

Exercise 3.3 (**Short-time existence of IMCF in a cone**). *Show that*

$$\begin{cases} \dfrac{\partial w}{\partial t} = \dfrac{1+|\nabla w|^2}{n-\left(\sigma^{ij}+\dfrac{\nabla^i w\nabla^j w}{1+|\nabla w|^2}\right)\nabla_{ij}w} & \text{in } M^n\times(0,T) \\ \nabla_\mu w = 0 & \text{on } \partial M^n\times(0,T) \\ w = \log u_0 & \text{on } M^n\times\{0\}. \end{cases}$$

has a unique solution at least for a short time. What else is needed to actually prove short-time existence of $(\mathrm{IMCF})_N$*?*

Exercise 3.4 (**Geometric estimate for star-shapedness under** $(\mathrm{IMCF})_N$**).**★ *Prove a lower bound for $f := \langle F,\nu\rangle$ under $(IMCF)_N$. Hint: Show that f satisfies the following parabolic problem:*

$$\begin{cases} \dfrac{\partial f}{\partial t} = \dfrac{1}{H^2}\Delta_g f + \dfrac{|A|^2}{H^2} f & \text{in } M_t^n\times(0,T) \\ {}^g\nabla_\mu f = f\,{}^{\Sigma^n}h_{\nu\nu} & \text{on } \partial M_t^n\times(0,T) \\ f(\cdot,0) = f_0 & \text{on } M_t^n \end{cases}$$

and use the maximum principle to conclude.

Exercise 3.5 (**Geometric estimate for the speed under** $(\mathrm{IMCF})_N$**).**★ *Prove an a priori estimate for $\dot w$. Hint: Show that $\dot w$ satisfies the following parabolic problem:*

$$\begin{cases} \dfrac{\partial \dot w}{\partial t} = \operatorname{div}_g\left(\dfrac{{}^g\nabla\dot w}{H^2}\right) - 2\dfrac{|{}^g\nabla \dot w|^2}{\dot w H^2} & \text{in } M_t^n\times(0,T) \\ {}^g\nabla_\mu \dot w = 0 & \text{on } \partial M_t^n\times(0,T) \\ \dot w(\cdot,0) = \dot w(\cdot,0) & \text{on } M_t^n. \end{cases}$$

and use the maximum principle and the previous estimates to conclude.