Introduction to nonlinear geometric PDEs

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Contents

Preface

These notes are the basis for an introductory lecture about geometric PDEs at ETH Zurich in the spring term 2013. The course is based on my diploma thesis about prescribed mean curvature problems, my PhD thesis about inverse mean curvature flow and many inspiring lectures by my former thesis advisor Gerhard Huisken.

Further information about the lecture can be found on the course web page:

www.math.ethz.ch/education/bachelor/lectures/hs2013/math/PDEs

If you have questions or comments please feel free to contact me:

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Part I.

Introduction and review of useful material

1. Introduction

1.1. Scope of the lecture

Geometric analysis is a field that has considerably grown over the last decades. Its goal is to answer questions that arise in geometry, topology, physics and many other sciences (e.g. in image processing) with the help of analytic tools. Usually the first task is to model the problem in terms of a (system of) PDE(s)[1]. Then existence and uniqueness is investigated using tools from PDE theory and/or the calculus of variations as well as functional analytic tools. Most of the time the geometric objects under consideration are not at all smooth. This requires the language of geometric measure theory to treat those problems.

The aim of the course is to give an introduction to the field of nonlinear geometric PDEs by discussing two typical classes of PDEs. For the first part of the course we will deal with nonlinear elliptic problems. In particular, we will look at the Dirichlet problem of prescribed mean curvature and the corresponding Neumann problem of capillary surfaces. In the second part we will investigate nonlinear parabolic PDEs. As an example we will discuss the evolution of surfaces under inverse mean curvature flow. We will prove short-time existence as well as convergence results and introduce the notion of weak solutions.

Prescribing the scalar mean curvature H of a hypersurface $M^n \subset \mathbb{R}^{n+1}$ which is given as the graph of a scalar function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ leads to the following equation[2]:

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = H(\cdot, u, Du) \quad \text{in } \Omega. \tag{1.1}$$

Together with a Dirichlet boundary condition $u = \phi$ on $\partial\Omega$ equation (1.1) is called the prescribed mean curvature problem. If we consider Neumann boundary conditions, i.e. if we prescribe the boundary contact angle the problem goes under the name capillary surface equation. During the first part of the course we will discuss existence and uniqueness of solutions to those problems. Note that if the denominator in (1.1) is replaced by one we obtain the Laplace operator. We will use the knowledge about linear second order elliptic PDEs together with a fixed point argument (or the method of continuity) and a priori estimates to prove existence for the corresponding nonlinear problems.

In the same way as the prescribed mean curvature equation resembles the Poisson equation, the evolution equation for the deformation of a hypersurface $M^n \subset N^{n+1}$ in time will resemble the heat equation. In our discussion we will focus on a deformation of hypersurfaces M^n along their inverse mean curvature. In terms of the embedding

[1] This step is not at all unique. People have come up with totally different successful models to answer exactly the same questions.

[2] We will discuss this in more detail in Section 2.1

$F : M^n \times [0,T] \to N^{n+1}$ the equation reads

$$\frac{\partial F}{\partial t} = \frac{1}{H}\nu \qquad F : M^n \times [0,T) \to (N^{n+1}, \overline{g}). \tag{1.2}$$

The similarity to the heat equation will become clear during the second part of the course. The investigation of (1.2) will lead us to the topic of nonlinear parabolic PDEs. We will analyze their well-posedness (i.e. short-time existence) as well as their long-time behavior. Finally we will also discuss the construction of weak solutions via the level set method. It turns out this procedure brings us back to a degenerate version of (1.1).

1.2. Accompanying books

The following books contain subjects which are relevant for the topics we will discuss during the semester. We will not follow a particular book. However, for the first part of the course the book by Gilbarg and Trudinger will be closest to the lecture notes. For the second part the book by Gerhardt might be the most relevant one.

Overview about the field of PDEs:

- Evans [17]

Elliptic PDEs of second order:

- Gilbarg, Trudinger [23]
- Ladyženskaja, Ural'ceva [37]

Parabolic PDEs of second order:

- Lieberman [40]
- Ladyženskaja, Solonnikov, Ural'ceva [38]

Maximum principle:

- Protter, Weinberger [53]

Minimal surfaces:

- Giusti [24]
- Dierkes, Hildebrand, Sauvigny [10]

Mean curvature flow and related flows:

- Gerhardt [22]
- Ecker [11]
- Ritoré, Sinestrari [54]
- Mantegazza, C. [41]

1.3. A historic survey

The field of geometric analysis is becoming more and more active during the last years. To give you an idea about the developments I made a brief survey in form of important results over the last 80 years. Note that the books and articles I cited here are not always written by the people who proved the result. If available I chose review articles which give an easy introduction into the topic.

1930: The Plateau problem
Solved independently by Douglas and Radó in 1930:

- Dierkes, Hildebrand, Sauvigny [10]

1979: The positive mass theorem
Proved by Schoen and Yau:

- Schoen (in Proc. of the Clay summer school 2001) [30]

1984: The Yamabe problem
Partial results by Trudinger, Aubin and others, finally solved by Schoen:

- Lee, Parker [39]
- Struwe [61]
- Bär [3]

1999: The Penrose inequality
Riemannian version proved by Huisken and Ilmanen and in a bit more general version two years later by Bray. The full Penrose inequality is still an open problem:

- Bray [5]

2002: The double bubble conjecture
Proved by Hutchings, Morgan, Ritoré and Ros.

- Morgan [47]

2003: The Poincaré conjecture
Proved by Perelman based on Hamilton's work on the Ricci flow:

- Ecker [12]
- Morgan, Tian [48]

2007: The differentiable sphere theorem
Proved by Brendle and Schoen:

- Brendle [6]

2012: The Lawson conjecture
Proved by Brendle:

- Brendle [7]

2012: The Wilmore conjecture
Proved by Marques and Neves:

- Marques, Neves [45]

Of course the list can not be complete. But if you have the feeling I missed something very important please let me know.

2. Review: Differential geometry

We will always work with orientable hypersurfaces which are either immersed or embedded in a Riemannian ambient manifold. For most of the things we will use the same notation as in the differential geometry class of Michael Eichmair [14].

2.1. Hypersurfaces in $\mathbb{R}^n$

For the discussion of the prescribed mean curvature problem and the capillary surface problem it will be sufficient to deal with embedded, graphical hypersurfaces in $\mathbb{R}^{n+1}$:

Let us consider a simple submanifold $M = \phi(U)$ of $\mathbb{R}^{n+1}$ where $U \subset \mathbb{R}^n$ is a chart domain[1]. The tangent space of M is defined as $TM := \bigcup_{p\in M} T_pM$ where T_pM is the tangent space of M at $p = \phi(x)$:

$$T_pM := \phi_*(T_xU) := \left\{ \left(\phi(x), D\phi\big|_x v\right) \mid (x,v) \in T_xU := U \times \mathbb{R}^n \right\}.$$

The normal space is defined as $NM := \bigcup_{p\in M} N_pM := \bigcup_{p\in M}(T_pM)^\perp$. In the case of an orientable hypersurface the normal space can be generated from a single normal vector ν.

The set of smooth tangent fields and normal fields are denoted by

$$\mathfrak{X}(M) := \Gamma(TM \to M) := \{X : M \to TM \mid X \text{ smooth}, \pi_{TM} \circ X = id_M\},$$
$$\Gamma(NM \to M) := \{\eta : M \to NM \mid \eta \text{ smooth}, \pi_{NM} \circ \eta = id_M\}$$

where $\pi_{TM} : TM \to M$ and $\pi_{NM} : NM \to M$ are the base point projections. A basis of $\Gamma(TM)$ is given by

$$\frac{\partial}{\partial x^i}\Big|_p := \phi_* \circ (x, e_i) \circ \phi^{-1} = \left(p, \frac{\partial \phi}{\partial x^i}\Big|_{\phi^{-1}(p)}\right).$$

The metric (or first fundamental form) of M is the symmetric, positive definite map

$$g : \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M) : (X,Y) \mapsto g(X,Y) := \langle X, Y\rangle.$$

For every p in M the map $g_p : T_pM \times T_pM \to \mathbb{R}$ is the restriction of the Euclidean inner product to T_pM. The matrix representation $[g]$ of g at $p = \phi(x)$ with respect to the basis mentioned above has the coefficients $g_{ij} := \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right\rangle$. The coefficients of the inverse matrix are denoted by g^{ij}.

The second fundamental form is the symmetric map

$$A : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \Gamma(NM \to M) : (X,Y) \mapsto A(X,Y) := -(D_XY)^\perp.$$

[1] Later we will use the word domain instead of chart domain and denote it by Ω instead of U.

where D is the (covariant) derivative on $T\mathbb{R}^{n+1}$. We obtain[2]

$$A(X,Y) = -\langle D_X Y, \nu\rangle \nu =: h(X,Y)\nu$$

The coefficients of the matrix $[h]$ with respect to the basis mentioned above are $h_{ij} = \left\langle -D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \nu \right\rangle$. The Eigenvalues of $[g]^{-1}[h]$ are called principle curvatures. Their sum is called mean curvature and their product is called Gaussian curvature.

Exercise I.1 (**Geometric meaning of the principal curvatures**)**.** Let M be a smooth surface in $\mathbb{R}^3$ with unit normal ν. Let $\varepsilon > 0$ and $\gamma : (-\varepsilon, \varepsilon) \to M$ be a smooth curve such that $\gamma(0) = p$ and $\gamma'(0) = v$ with $|v| = 1$. We define the curvature of M at p in direction v as

$$k_p : T_pM \to \mathbb{R} : v \mapsto k_p(v) := \langle \gamma''(0), \nu \rangle.$$

Note that by Meusnier's theorem k_p is well defined. Answer the following questions

(i) Why does it make sense to call k_p the curvature of M at p in dir. v?

(ii) What is the relation between k_p and h?

(iii) What are the critical values of k_p in terms of h and g?

The tangential covariant derivative on TM is defined as

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M) : (X,Y) \mapsto \nabla_X Y := (D_X Y)^\top$$

where $(\nabla_X Y)(p) := \nabla_{X(p)} Y = (D_{X(p)} Y)^\top$. For computations the Leibniz rule and metric compatibility are important tools

$$\nabla_X fY = (Xf)Y + f\nabla_X Y, \qquad Zg(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$$

for all $X, Y, Z \in \Gamma(TM)$ and $f \in C^\infty(M)$. Recall that

$$Xf = X^i \frac{\partial}{\partial x^i} f := X^i \frac{\partial(f \circ \phi)}{\partial x^i}\Big|_{\phi^{-1}}.$$

where $\frac{\partial}{\partial x^i}$ is used as a symbol for the basis element as well as for the actual partial derivative in $\mathbb{R}^n$.

The functions Γ^k_{ij} such that $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k}$ are called Christoffel symbols. It is not difficult to verify that

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right).$$

It follows that for $X = X^i \frac{\partial}{\partial x^i}, Y = Y^j \frac{\partial}{\partial x^j}$ we have

$$\nabla_X Y = X^i \left(\frac{\partial Y^k}{\partial x^i} + Y^j \Gamma^k_{ij} \right) \frac{\partial}{\partial x^k}.$$

[2]Note that the sign of h depends on the choice of normal ν.

Based on the rules that taking covariant derivatives commutes with contractions and that $\nabla_X(S\otimes T) = (\nabla_X S)\otimes T + S\otimes(\nabla_X T)$ we obtain connections on the associated vector bundles. For example, for differential one forms we get $(\nabla_X\omega)Y = X\omega(Y) - \omega(\nabla_X Y)$. In coordinates, i.e. $X = X^i\frac{\partial}{\partial x^i}, \omega = \omega_k dx^k$ this yields

$$\nabla_X\omega = X^i\left(\frac{\partial\omega_k}{\partial x^i} - \omega_j\Gamma^j_{ik}\right)dx^k.$$

The tangential gradient of a function $f \in C^\infty(M)$ is defined to be the unique tangent field[3] $\operatorname{grad}_M f$ such that $g(\operatorname{grad}_M f, X) = df(X) = Xf$ for all X in $\Gamma(TM)$. We obtain the formulae

$$\operatorname{grad}_M f = g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j} = \operatorname{grad}_{\mathbb{R}^{n+1}} f - \langle\operatorname{grad}_{\mathbb{R}^{n+1}} f, \nu\rangle\,\nu = \sum_{i=1}^m (D_{\tau_i} f)\tau_i$$

where we used an orthonormal frame (τ_i) of TM in the last equality.

The tangential divergence of a tangent field X can be defined via the contraction of ∇X. We can write

$$\operatorname{div}_M X := \sum_{i=1}^n\left(\frac{\partial X^i}{\partial x^i} + \Gamma^i_{ij}X^j\right) = \operatorname{div}_{\mathbb{R}^{n+1}} X - \langle D_\nu X, \nu\rangle = \sum_{i=1}^m\langle D_{\tau_i}X, \tau_i\rangle$$

using once more an orthonormal frame of TM for the last expression. Note that the last two equalities also make sense for vector fields which are not necessarily tangential.

In the special case $\omega = df = \frac{\partial f}{\partial x^i}dx^i$ we obtain the Hessian of f:

$$(\operatorname{Hess}_M f)(X,Y) = (\nabla df)(X,Y) = (\nabla_X df)(Y) = X^iY^j\left(\frac{\partial^2 f}{\partial x^i\partial x^j} - \frac{\partial f}{\partial x^k}\Gamma^k_{ij}\right).$$

Finally, the Laplace Beltrami operator is defined as $\Delta_M f := \operatorname{div}_M(\operatorname{grad}_M f)$. One can compute that

$$\Delta_M f = g^{ij}\left(\frac{\partial^2 f}{\partial x^i\partial x^j} - \frac{\partial f}{\partial x^k}\Gamma^k_{ij}\right) = g^{ij}(\operatorname{Hess}_M f)_{ij}.$$

Thus it is the contraction of the Hessian with respect to the metric.

Exercise I.2 (**Graphical hypersurfaces in** $\mathbb{R}^n$**).** Suppose that M is a hypersurface in $\mathbb{R}^{n+1}$ which is given as a graph over a chart domain $U \subset \mathbb{R}^n$: $M = \operatorname{graph} u$. Verify the following formulae:

$$g_{ij} = \delta_{ij} + D_iuD_ju, \qquad g^{ij} = \delta^{ij} - \frac{D^iuD^ju}{1+|Du|^2}, \qquad \Gamma^k_{ij} = \frac{D_{ij}uD^ku}{1+|Du|^2},$$

$$\nu = \frac{\pm 1}{\sqrt{1+|Du|^2}}(-Du, 1), \qquad h_{ij} = \frac{\mp D_{ij}u}{\sqrt{1+|Du|^2}},$$

$$H = \frac{\mp 1}{\sqrt{1+|Du|^2}}\left(\delta^{ij} - \frac{D^iuD^ju}{1+|Du|^2}\right)D_{ij}u = \operatorname{div}_{\mathbb{R}^n}\left(\frac{\mp Du}{\sqrt{1+|Du|^2}}\right) = \operatorname{div}_{\mathbb{R}^{n+1}}(\nu),$$

$$\mathbf{H} := -H\nu = \operatorname{div}_{\mathbb{R}^n}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)\frac{1}{\sqrt{1+|Du|^2}}\begin{pmatrix}-Du\\ 1\end{pmatrix}.$$

[3]Sometimes people just write ∇f. In the analytical literature such as [23] one also finds the symbol δf.

Remark 2.1 (Classical problems). A natural question is, weather for a given function H and given boundary values ϕ on $\partial\Omega$ there exists a graph of a function $u : \Omega \to \mathbb{R}$ which has mean curvature H and attains the boundary values ϕ, i.e. a solution to the prescribed mean curvature problem

$$\begin{cases} \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = H(\cdot, u, Du) \quad \text{in } \Omega \\ u = \phi \quad \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

If at the boundary we prescribed the contact angle instead of the height we obtain the so called capillary surface problem

$$\begin{cases} \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = H(\cdot, u, Du) \quad \text{in } \Omega \\ \frac{D_\gamma u}{\sqrt{1+|Du|^2}} = \beta \quad \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

where β is the cosine of the contact angle and γ is the outward unit normal to $\partial\Omega$. One can regard various modifications of these problems. For example, one can replace the mean curvature by the Gaussian curvature, i.e.

$$\begin{cases} \frac{\det(D^2u)}{(1+|Du|^2)^{\frac{n+2}{2}}} = K(\cdot, u, Du) \quad \text{in } \Omega \\ u = \phi \quad \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

and similar for the Neumann problem. Furthermore, one can consider different ambient spaces, e.g. hyperbolic space or Minkowski space or general Riemannian manifolds. One way to find solutions to these static problems is to consider the corresponding evolution equation and to investigate what happens in the limit as $t \to \infty$, e.g.

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = H(\cdot, u, Du) \quad \text{in } \Omega \times (0,T) \\ u = \phi \quad \text{on } \partial\Omega \times (0,T) \\ u = u_0 \quad \text{on } \Omega \times \{0\}. \end{cases} \tag{2.4}$$

The corresponding linear model problem would then be the heat equation. These parabolic problems are also interesting in its own as they might reveal topological information about the evolving surface. A totally different application would be to apply those flows to do a noise reduction in image processing.

Exercise I.3 (Explicit computation). Compute the principal curvatures, the mean curvature and the Gaussian curvature of the graph of the function $u : B_2(0) \to \mathbb{R} : (x,y) \mapsto x^2 - y^2$ at the origin.

Exercise I.4 (Counter examples). (i) Let $R > 0$ and $\Omega = B_R(0) \subset \mathbb{R}^n$. Is there always a function $u : \Omega \to \mathbb{R}$ with $u = 0$ on $\partial\Omega$ such that the graph of u has constant mean curvature $H = c$?

(ii) Let $0 < R_1 < R_2 < \infty$ and $\Omega = B_{R_2}(0) \setminus \overline{B_{R_1}(0)}$. Is there always a function $u : \overline{\Omega} \to \mathbb{R}$ with $u = 0$ on $\{|x| = R_2\}$ and $u = L > 0$ on $\{|x| = R_1\}$ such that the graph of u has zero mean curvature? Hint: Try to find an explicit solution.

Exercise I.5 (Flat v.s. harmonic v.s. minimal). Let $\Omega = B_2(0) \setminus \overline{B_1(0)}$ and the boundary conditions be $u = 0$ on $\{|x| = 2\}$ and $u = 1$ on $\{|x| = 1\}$. Compare the surface area of the graphical minimal surface graph v_m to that of the truncated cone, i.e. to graph v_c where $v_c(x) := 2 - |x|$ as well as to graph v_h where v_h is the corresponding harmonic function, i.e. the function satisfying $\Delta v_h = 0$ in Ω together with the same boundary values.

2.2. Isometric immersions

Let $(N, \overline{g})$ be a Riemannian manifold of dimension n. Let M be a smooth manifold of dimension $m \leq n$ and $\phi : M \to N$ an immersion, i.e. a smooth map, such that for all $p \in M$ the push-forward $\phi_\star : T_pM \to T_{\phi(p)}N$ is injective (so that in charts we have $\operatorname{rank}[D\phi] = \dim M$ everywhere). If additionally ϕ is a homeomorphism onto its image it is called an embedding. We define the pull-back metric on TM via

$$g_p(v, w) := (\phi^\star \overline{g})_p\,(v, w) := \overline{g}_{\phi(p)}(\phi_\star v, \phi_\star w) \qquad \forall\; v, w \in T_pM.$$

With respect to that metric the map $\phi : (M, g) \to (N, \overline{g})$ is an isometric immersion. We can also pull back the tangent bundle $TN \to N$ to obtain the bundle $\phi^\star TN \to M$ where

$$\phi^\star TN := \bigcup_{p \in M} \{p\} \times T_{\phi(p)}N$$

with the tangent space and normal space as subspaces:

$$TM_\phi := \bigcup_{p \in M} \{p\} \times \phi_\star(T_pM), \qquad NM_\phi := \bigcup_{p \in M} \{p\} \times (\phi_\star(T_pM))^\perp .$$

Note that we can identify TM and TM_ϕ via the isometry $v \mapsto (\pi(v), \phi_\star v)$. Using the projections $^\perp : TN \to NM_\phi$ and $^\top : TN \to TM$ we have the decomposition

$$V = \phi_\star \circ (V^\top) + V^\perp \qquad \forall\; V \in TN.$$

On $(N, \overline{g})$ there exists a unique covariant derivative (also called connection) which is compatible with $\overline{g}$ and torsion free, i.e.

$$Z\overline{g}(V, W) = \overline{g}(\overline{\nabla}_Z V, W) + \overline{g}(V, \overline{\nabla}_Z W), \qquad \overline{\nabla}_V W - \overline{\nabla}_W V = [V, W].$$

This covariant derivative $\overline{\nabla}$ is called Levi-Civita connection of $(N, \overline{g})$. It can be shown that the connection

$$\nabla_X Y := \left({}^\phi\overline{\nabla}_X \phi_\star \circ Y\right)^\top \qquad X, Y \in \mathfrak{X}(M)$$

is the Levi-Civita connection of (M, g). Here ${}^\phi\overline{\nabla}$ is the pull-back connection, i.e. the unique connection ${}^\phi\overline{\nabla} : \mathfrak{X}(M) \times \Gamma(\phi^\star TN \to M) \to \Gamma(\phi^\star TN \to M)$ which satisfies the naturality condition

$${}^\phi\overline{\nabla}_v \phi^\star \eta = \overline{\nabla}_{\phi_\star v}\eta \qquad \forall\; v \in TM, \quad \forall\; \eta \in \Gamma(TN \to N).$$

The normal part of the connection is called the second fundamental form:

$$A(X,Y) := -\left({}^{\phi}\overline{\nabla}_X \phi_\star \circ Y\right)^{\perp} \qquad X, Y \in \mathfrak{X}(M).$$

Thus, we can write $-A(X,Y) = {}^{\phi}\overline{\nabla}_X \phi_\star \circ Y - \phi_\star \circ \nabla_X Y$. With respect to a basis we have

$$\begin{aligned}
-A_{ij} &= {}^{\phi}\overline{\nabla}_{\frac{\partial}{\partial x^i}} \left[\left(\phi_\star \circ \frac{\partial}{\partial x^j}\right)^{\alpha} \phi^\star \frac{\partial}{\partial q^\alpha}\right] - \phi_\star \circ \left(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}\right) \\
&= \frac{\partial}{\partial x^i}\left(\phi_\star \circ \frac{\partial}{\partial x^j}\right)^{\alpha} \frac{\partial}{\partial q^\alpha}\Big|_{\phi} + \left(\phi_\star \circ \frac{\partial}{\partial x^j}\right)^{\alpha} {}^{\phi}\overline{\nabla}_{\frac{\partial}{\partial x^i}} \phi^\star \frac{\partial}{\partial q^\alpha} - \Gamma^k_{ij} \phi_\star \circ \frac{\partial}{\partial x^k} \\
&= \frac{\partial}{\partial x^i}\left(\phi_\star \circ \frac{\partial}{\partial x^j}\right)^{\alpha} \frac{\partial}{\partial q^\alpha}\Big|_{\phi} + \left(\phi_\star \circ \frac{\partial}{\partial x^j}\right)^{\alpha} \left(\phi_\star \circ \frac{\partial}{\partial x^i}\right)^{\beta} {}^{\phi}\overline{\Gamma}^{\gamma}_{\alpha\beta} \phi^\star \frac{\partial}{\partial q^\gamma} - \Gamma^k_{ij} \phi_\star \circ \frac{\partial}{\partial x^k} \\
&= \frac{\partial^2 \phi}{\partial x^i \partial x^j} + \frac{\partial \phi^\alpha}{\partial x^i} \frac{\partial \phi^\beta}{\partial x^j} {}^{\phi}\overline{\Gamma}^{\gamma}_{\alpha\beta} \frac{\partial}{\partial p^\gamma}\Big|_{\phi} - \Gamma^k_{ij} \frac{\partial \phi}{\partial x^k}.
\end{aligned}$$

Note that the last equality is just symbolic unless $N = \mathbb{R}^n$. In that case the $\partial/\partial x^r$ can be read as partial derivatives and $\overline{\Gamma} \equiv 0$.

2.3. First variation of area

Proposition 2.2 (First variation of area). *Let $M \subset \mathbb{R}^n$ be a smooth m-dimensional submanifold. Let $O \subset \mathbb{R}^n$ be open such that $M \cap O \neq \emptyset$. We consider the deformation of M under a family of diffeomorphisms*

$$\Phi : (-\varepsilon, \varepsilon) \times O \to O : (t,p) \mapsto \Phi(t,p) =: \Phi_t(p)$$

satisfying for some compact set $K \subset O$

$$\Phi(0,\cdot) = Id \quad \text{in } O, \qquad \Phi(t,\cdot) = Id \quad \text{in } O \setminus K.$$

Then for the first variation of area we obtain

$$\frac{d}{dt}\Big|_{t=0} \operatorname{area}(\Phi_t(M)) = \int_M \operatorname{div}_M \left(\frac{d}{dt}\Big|_{t=0} \Phi_t\right) \, d\mu.$$

For the prove we will use the following formula.

Exercise I.6 (Derivative of the determinant). Let $\varepsilon > 0$. Suppose that

$$B \in C^1\Big((t_0 - \varepsilon, t_0 + \varepsilon), \mathbb{R}^{n \times n}\Big)$$

and that $B(t_0)$ is invertible. Show that

$$\frac{d}{dt}\Big|_{t=t_0} \det B(t) = \det B(t_0) \operatorname{tr}\left(B^{-1}(t_0) \frac{d}{dt}\Big|_{t=t_0} B(t)\right).$$

Proof of Proposition 2.2. The area formula tells us that

$$\operatorname{area}(\Phi_t(M)) = \int_M \operatorname{Jac}((\Phi_t)_\star)\, d\mu = \int_M \sqrt{\det([(\Phi_t)_\star]^\top [(\Phi_t)_\star])}\, d\mu.$$

Therefore, it remains to compute the derivative of the Jacobian. To simplify our computation we write

$$\Phi_t(p) = p + tX(p) + O(t^2), \qquad X := \frac{d}{dt}\Big|_{t=0} \Phi_t.$$

For the push forward of Φ_t we obtain

$$(\Phi_t)_\star : TM \to T\mathbb{R}^n : \tau_k \mapsto (\Phi_t)_\star \tau_k = \tau_k + tD_{\tau_k} X + O(t^2).$$

The matrix representation of $(\Phi_t)_\star$ with respect to an orthonormal basis $(\tau_i)_{1\le i\le m}$ of TM and the standard basis $(e_j)_{1\le j\le n+1}$ of $T\mathbb{R}^{n+1}$ is given by

$$[(\Phi_t)_\star]_{ij} = \tau_i^j + tD_{\tau_i} X^j + O(t^2).$$

Using the formula for the derivative of the determinant we can compute that

$$\begin{aligned}
&\frac{d}{dt}\Big|_{t=0} \sqrt{\det([(\Phi_t)_\star]^\top [(\Phi_t)_\star])} \\
&\quad = \frac{d}{dt}\Big|_{t=0} \sqrt{\det(\delta_{ij} + t\langle \tau_i, D_{\tau_j} X\rangle + t\langle D_{\tau_i} X, \tau_j\rangle + O(t^2))} \\
&\quad = \frac{1}{2} \operatorname{tr} \Big(\langle \tau_i, D_{\tau_j} X\rangle + \langle D_{\tau_i} X, \tau_j\rangle \Big) \\
&\quad = \operatorname{div}_M X
\end{aligned}$$

which proves the result. □

Remark 2.3. To derive a formula for the second variation of area one can proceed in a similar way. For the details see [56], Chapter 2.

Corollary 2.4. *Let us define* $X := \frac{d}{dt}\Big|_{t=0} \Phi_t$ *then*

$$\frac{d}{dt}\Big|_{t=0} \operatorname{area}(\Phi_t(M)) = \int_M \operatorname{div}_M X\, d\mu = + \int_M H\langle \nu, X\rangle\, d\mu + \int_{\partial M} \langle \mu, X\rangle\, d\sigma$$

where ν *is the unit normal of* M *and* μ *is the outward unit conormal of* ∂M*, i.e. normal to* ∂M*, tangent to* M *and pointing away from* M*.*

Proof. Proposition 2.2 implies the first equality. To verify the second equality we choose an orthonormal frame $(\tau_i)_{i\in\mathbb{N}}$ of TM and compute

$$\begin{aligned}
\int_M \operatorname{div}_M X\, d\mu &= \int_M \operatorname{div}_M X^\top\, d\mu + \int_M \operatorname{div}_M X^\perp\, d\mu \\
&= \int_{\partial M} \Big\langle \mu, X^\top \Big\rangle\, d\sigma + \int_M \Big\langle D_{\tau_i} X^\perp, \tau_i \Big\rangle\, d\mu \\
&= \int_{\partial M} \langle \mu, X\rangle\, d\sigma + \int_M \tau_i \Big(\Big\langle X^\perp, \tau_i \Big\rangle \Big)\, d\mu - \int_M \Big\langle (D_{\tau_i}\tau_i)^\perp, X \Big\rangle\, d\mu \\
&= \int_{\partial M} \langle \mu, X\rangle\, d\sigma - \int_M \Big\langle \langle D_{\tau_i}\tau_i, \nu\rangle \nu, X \Big\rangle\, d\mu
\end{aligned}$$

which is the desired result. □

Remark 2.5 (Minimal surfaces). Observe that if X is normal to M then the first derivative of area vanishes exactly for surfaces of zero mean curvature. Even though $H = 0$ surfaces are just critical points of the area functional they are often called minimal surfaces. Try to picture what happens with M if X has a non-vanishing tangential component.

Remark 2.6 (Variational approach). Note that in the special case of graphical hypersurfaces in $\mathbb{R}^n$ minimizing area means

$$I(u) := \int_\Omega \sqrt{1 + |Du(x)|^2} dx \;\to\; \min .$$

The corresponding Euler-Lagrange equation is exactly the minimal surface equation. Furthermore, the analogous linear problem is the minimization of the Dirichlet energy, i.e.

$$I(u) := \int_\Omega |Du(x)|^2 dx \;\to\; \min$$

whose Euler-Lagrange equations is $\Delta u = 0$.

Remark 2.7 (Capillary surfaces in gravitational fields). If in addition to area (which up to a constant equals surface tension) we also take gravity into account as well as the adhesion forces at the boundary, physical considerations lead to minimizing

$$I(u) := \int_\Omega \sqrt{1 + |Du(x)|^2} dx + \kappa \int_\Omega u(x)^2 dx - \int_{\partial\Omega} \beta u(x) ds \;\to\; \min .$$

The corresponding elliptic problem is the capillary surface problem, where $H(\cdot, u, Du) = \kappa u$.

3. Review: Linear PDEs of second order

3.1. Elliptic PDEs in Hölder spaces

Definition 3.1 (**Linear elliptic operators**)**.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $u \in C^2(\overline{\Omega})$. Suppose that $a^{ij}, b^k, c \in C^0(\overline{\Omega})$ and that the matrix $[a^{ij}]$ is symmetric. The differential operator L defined by

$$Lu := a^{ij}(x)D_{ij}u \ + \ b^k(x)D_k u \ + \ c(x)u$$

is called elliptic in Ω if the matrix $[a^{ij}(x)]$ is positive definite for all x in Ω. In this case

$$0 < \ \lambda(x) \ \leq \ a^{ij}(x)\xi_i\xi_j \ \leq \ \Lambda(x) \qquad \forall \ \ \xi \in \mathbb{S}^n, \quad \forall \ x \in \Omega \tag{3.1}$$

where $\lambda(x)$ and $\Lambda(x)$ are the smallest and largest Eigenvalues of $[a^{ij}(x)]$. Furthermore, L is called uniformly elliptic in Ω if there exist $\lambda_{\min}$ and $\Lambda_{\max}$ such that

$$0 < \lambda_{\min} \ \leq \ a^{ij}(x)\xi_i\xi_j \ \leq \ \Lambda_{\max} \ < \infty \qquad \forall \ \ \xi \in \mathbb{S}, \quad \forall \ x \in \Omega. \tag{3.2}$$

Theorem 3.2 (**Maximum principle**)**.** *Let L be uniformly elliptic in the bounded domain Ω and $c \leq 0$. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$. If $Lu \geq f$ then*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + c \sup_{\Omega} \left(\frac{|f^-|}{\lambda} \right).$$

If $Lu = f$ then

$$\sup_{\Omega} |u| \leq \sup_{\partial\Omega} |u| + c \sup_{\Omega} \left(\frac{|f|}{\lambda} \right).$$

In both cases $c = c(\operatorname{diam}\Omega, \sup |b|/\lambda)$.

Proof. See [23], Theorem 3.7. □

The following comparison principle is a useful consequence.

Corollary 3.3. *Let Ω and L be as above. Suppose that $u, v \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ satisfy $Lu \geq Lv$ in Ω and $u \leq v$ on $\partial\Omega$. Then $u \leq v$ in $\overline{\Omega}$.*

Proof. Apply Theorem 3.2 to $w := u - v$. □

Before we can state the existence theorem for linear elliptic equations of second order we want to recall the definition of Hölder spaces.

Definition 3.4 (Hölder spaces). Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $0 < \alpha < 1$. A function $f : \Omega \to \mathbb{R}$ is called α-Hölder continuous in x_0 if

$$[f]_{\alpha,\{x_0\}} := \sup_{x\in\Omega} \frac{|f(x) - f(x_0)|}{|x - x_0|^\alpha} < \infty.$$

We say that f is uniformly α-Hölder continuous in Ω if

$$[f]_{\alpha,\Omega} := \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty.$$

The spaces

$$C^{k,\alpha}(\overline{\Omega}) := \left\{ f \in C^k(\overline{\Omega}) \;\middle|\; D^\beta f \text{ unif. } \alpha\text{-Hölder cont. in } \Omega,\ \forall\, \beta \in \mathbb{N}^n,\ |\beta| = k \right\}$$

equipped with the norm

$$\|f\|_{C^{k,\alpha}(\overline{\Omega})} := \|f\|_{C^k(\overline{\Omega})} + \sum_{|\beta|=k} [D^\beta f]_{\alpha,\Omega}$$

are Banach spaces. If $\alpha = 1$ we say Lipschitz instead of 1-Hölder.

Lemma 3.5. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded Lipschitz domain. Let $k_1, k_2 \geq 0$ and $0 \leq \alpha_1, \alpha_2 \leq 1$. If $k_1 + \alpha_1 > k_2 + \alpha_2$ then the inclusion of $C^{k_1,\alpha_1}(\overline{\Omega})$ into $C^{k_2,\alpha_2}(\overline{\Omega})$ is compact.*

Proof. See [1], Section 8.6. □

Now we are ready to quote a classical existence theorem for linear Dirichlet problems.

Theorem 3.6 (Existence for the linear Dirichlet problem). *Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{2,\alpha}$-domain. Let L be an uniformly elliptic operator with coefficients in $C^{0,\alpha}(\overline{\Omega})$ and $c \leq 0$. Furhtermore, assume that $f \in C^{0,\alpha}(\overline{\Omega})$ and $\phi \in C^{2,\alpha}(\overline{\Omega})$. Then the Dirichlet problem*

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \phi & \text{on } \partial\Omega \end{cases}$$

has a unique solution $u \in C^{2,\alpha}(\overline{\Omega})$ satisfying

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \leq C \left(\|u\|_{C^0(\overline{\Omega})} + \|\phi\|_{C^{2,\alpha}(\overline{\Omega})} + \|f\|_{C^{0,\alpha}(\overline{\Omega})} \right)$$

where $C = C\left(n, \alpha, \Omega, \lambda_{\min}, \|a^{ij}\|_{C^{0,\alpha}(\overline{\Omega})}, \|b^i\|_{C^{0,\alpha}(\overline{\Omega})}, \|c\|_{C^{0,\alpha}(\overline{\Omega})}\right).$

Proof. See [23], Theorem 6.6 and Theorem 6.14. □

Remark 3.7. The condition $c \leq 0$ is only needed for the existence and uniqueness statement but not for the estimate. In the case that $c \leq 0$ is not satisfied existence and uniqueness still hold as long as the homogeneous problem $Lu = 0$ in Ω, $u = 0$ on $\partial\Omega$ has only the zero solution. That is (one part of) the Fredholm alternative (see [23], Theorem 6.15).

If the data of the problem are more regular then also the solution possesses more regularity.

Theorem 3.8 (**Interior regularity**). *Let $k \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ a bounded domain. Let L be a linear, uniformly elliptic operator with coefficients $a^{ij}, b^i, c \in C^{k,\alpha}(\Omega)$. Furthermore, assume that $f \in C^{k,\alpha}(\Omega)$ and that $u \in C^2(\Omega)$ satisfies $Lu = f$. Then $u \in C^{k+2,\alpha}(\Omega)$.*

Proof. See [23], Theorem 6.17. □

Theorem 3.9 (**Global regularity**). *Let $k \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ a bounded $C^{k+2,\alpha}$-domain. Let L be a linear, uniformly elliptic operator with coefficients $a^{ij}, b^i, c \in C^{k,\alpha}(\overline{\Omega})$. Furthermore, assume that $f \in C^{k,\alpha}(\overline{\Omega})$ and $\phi \in C^{k+2,\alpha}(\overline{\Omega})$. If $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ satisfies $Lu = f$ in Ω and $u = \phi$ on $\partial\Omega$. Then $u \in C^{k+2,\alpha}(\overline{\Omega})$.*

Proof. See [23], Theorem 6.19. □

Finally, we also want to mention the corresponding result for the oblique derivative problem. It includes the Neumann problem as a particular case.

Theorem 3.10 (**Existence for the linear oblique Derivative problem**). *Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{2,\alpha}$-domain. Let L be a linear, uniformly elliptic operator with coefficients in $C^{0,\alpha}(\overline{\Omega})$ and $c \leq 0$. Furhtermore, assume that $f \in C^{0,\alpha}(\overline{\Omega})$ and $\gamma, \beta, \phi \in C^{1,\alpha}(\overline{\Omega})$. If*

$$\gamma\langle \beta, \nu\rangle > 0 \qquad (\nu \text{ exterior unit normal of } \partial\Omega)$$

or

$$\langle \beta, \nu\rangle > 0, \quad \gamma \geq 0 \quad \text{and} \quad \text{either } c \neq 0 \text{ or } \gamma \neq 0,$$

then the oblique derivative problem

$$\begin{cases} Lu = f & \text{in } \Omega \\ \gamma u + D_\beta u = \phi & \text{on } \partial\Omega \end{cases}$$

has a unique solution $u \in C^{2,\alpha}(\overline{\Omega})$ satisfying

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \;\leq\; C\left(\|u\|_{C^0(\overline{\Omega})} + \|\phi\|_{C^{1,\alpha}(\overline{\Omega})} + \|f\|_{C^{0,\alpha}(\overline{\Omega})}\right) \tag{3.3}$$

where

$$C = C\Big(n, \alpha, \Omega, \lambda_{\min}, \|a^{ij}\|_{C^{0,\alpha}(\overline{\Omega})}, \|b^i\|_{C^{0,\alpha}(\overline{\Omega})}, \|c\|_{C^{0,\alpha}(\overline{\Omega})}, \\ \|\gamma\|_{C^{1,\alpha}(\overline{\Omega})}, \|\beta_i\|_{C^{1,\alpha}(\overline{\Omega})}, \langle \beta, \nu\rangle\Big).$$

Proof. See [23], Theorem 6.30, Theorem 6.31 and the following remarks about the Fredholm alternative. □

Exercise I.7 (**Understanding Hölder spaces**). (i) Can you find a function which is in $C^{0,\alpha}(\overline{\Omega})$ but not in $C^{0,\alpha+\varepsilon}(\overline{\Omega})$ for $\varepsilon > 0$? Can you find a function that is in $C^{1,1}(\overline{\Omega})$ but not in $C^2(\overline{\Omega})$?

(ii) Can you find a domain $\Omega \subset \mathbb{R}^2$ and a function $u \in C^1(\overline{\Omega})$ which is not in $C^{0,3/4}(\overline{\Omega})$?

(iii) Why do we need to work with Hölder spaces $C^{k,\alpha}(\overline{\Omega})$ instead of the easier $C^k(\overline{\Omega})$ spaces?

3.2. Elliptic PDEs in Sobolev spaces

Definition 3.11 (**Sobolev spaces**). Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $\alpha \in \mathbb{N}^n$ a multi index. A function $v \in L^1_{loc}(\Omega)$ is called weak α-derivative of $u \in L^1_{loc}(\Omega)$ if

$$\int_\Omega \psi v \, dx = (-1)^{|\alpha|} \int_\Omega u D^\alpha \psi \, dx \qquad \forall \; \psi \in C_0^{|\alpha|}(\Omega).$$

We write $v = D^\alpha u$ weakly. For $k \in \mathbb{N}$ and $1 \leq p \leq \infty$ the sets

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) \mid D^\alpha u \in L^p(\Omega) \; for \; |\alpha| \leq k\}$$

are called Sobolev spaces. Equipped with the norms

$$\|u\|_{W^{k,p}(\Omega)} := \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(\Omega)}, \qquad 1 \leq p < \infty$$

$$\|u\|_{W^{k,\infty}(\Omega)} := \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^\infty(\Omega)}, \qquad p = \infty$$

they are Banach spaces. The spaces $H^k(\Omega) := W^{k,2}(\Omega)$ are even Hilbert spaces. Furthermore, we extend the notion of functions having zero boundary values by defining

$$W_0^{k,p}(\Omega) := \overline{C_0^k(\overline{\Omega})}^{\|\cdot\|_{W^{k,p}(\Omega)}}.$$

To get a better understanding of these spaces let us have a look at the following theorem.

Theorem 3.12 (**Trace operator**). *Let $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^n$ be a bound domain with C^1-boundary. There exists a bounded linear operator $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ called the trace operator defined via*

$$Tu = u|_{\partial\Omega} \qquad for \qquad u \in C^0(\overline{\Omega}) \cap W^{1,p}(\Omega).$$

If $u \in W^{1,p}(\Omega)$ then $u \in W_0^{1,p}(\Omega)$ if and only if $Tu|_{\partial\Omega} = 0$.

Proof. See [17], Section 5.5, Theorem 1 and Theorem 2. □

Lemma 3.13. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $u \in W^{1,p}(\Omega)$. Then*

$$u^+ := \max\{u, 0\}, u^- := \min\{u, 0\}, |u| \in W^{1,p}(\Omega).$$

Furthermore, $Du = 0$ on any set where u is constant.

Proof. See [23], Lemma 7.6. and Lemma 7.7. □

Theorem 3.14 (**Poincaré type inequalities**). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $1 \leq p < n$ and $u \in W_0^{1,p}(\Omega)$. Then*

$$\|u\|_{L^q(\Omega)} \leq C(p, q, n, \Omega) \|Du\|_{L^p(\Omega)} \qquad \forall \; q \in \left[1, \frac{np}{n-p}\right].$$

If $p = 1$ we can choose the constant to be $c = (n\omega^{1/n})^{-1}$. Another special case is

$$\|u\|_{L^p(\Omega)} \leq \left(\frac{|\Omega|}{\omega_n}\right)^{1/n} \|Du\|_{L^p(\Omega)}.$$

Proof. See [17], Section 5.6, Theorem 3 and [23], inequality (7.44). □

An important relation between the Hölder spaces and the Sobolev spaces is given by (one of) the Sobolev embedding theorem(s).

Theorem 3.15 (Sobolev embedding theorem). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Let $p \in [1, \infty)$ and $u \in W^{k,p}(\Omega)$. If*

$$0 \leq m < k - \frac{n}{p} < m + 1$$

then $u \in C^{m,k-\frac{n}{p}-m}(\overline{\Omega})$. For smaller Hölder coefficients the embedding is even compact. Note that the regularity assumption for $\partial\Omega$ can be dropped if we consider the space $W_0^{k,p}(\Omega)$ instead.

Proof. See for example [23], Chapter 7, Theorem 7.26 or [17], Chapter 5, Section 6, Theorem 6. □

Theorem 3.16 (Morrey's estimate). *Let $u \in W^{1,1}(\Omega)$. If there exists some $\alpha \in (0, 1)$ and $K > 0$ such that*

$$\int_{B_R} |Du| dx \leq K R^{n-1+\alpha}, \qquad \forall\ B_R \subset \Omega$$

Then $u \in C^{0,\alpha}(\Omega)$ with $[Du]_{\alpha,\Omega} \leq c(n, \alpha, K)$. If $\Omega = \tilde{\Omega} \cap \{x_n > 0\}$ for some domain $\tilde{\Omega} \subset \mathbb{R}^n$ and the above inequality holds for all $B_R \subset \tilde{\Omega}$ then $u \in C^{0,\alpha}(\overline{\Omega} \cap \tilde{\Omega})$.

Proof. See [23], Theorem 7.19. □

Remark 3.17. The last part of the theorem will be particularly useful for local estimates near a flattened boundary.

Next we want to recall the Hölder continuity of weak solutions.

Definition 3.18 (Weak solutions). Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We consider the linear operator

$$Lu := D_i(a^{ij} D_j u) + b^k D_k u + cu$$

with coefficients $a^{ij}, b^k, c \in C^0(\overline{\Omega})$ and such that $[a^{ij}]$ satisfies (3.2). Let $g, f^i \in L^1(\Omega)$. If $u \in W^{1,2}(\Omega)$ satisfies

$$\int_\Omega \left[\left(a^{ij} D_j u - f^i \right) D_j \xi - \left(b^k D_k u + cu - g \right) \xi \right] = 0 \qquad \forall\ \xi \in C_0^1(\Omega)$$

we say that u is a <u>weak solution</u> of $Lu = g + D_i f^i$.

Remark 3.19. Note that weak solutions can be defined for more general operators and under weaker conditions on the coefficients. Note also that the integral equality remains true for test functions $\xi \in W_0^{1,2}(\Omega)$.

Theorem 3.20 (Maximum principle). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and L as in Definition 3.18. Let $u \in C^0(\overline{\Omega}) \cap W^{1,2}(\Omega)$ satisfy $Lu \geq 0$ in a weak sense. If $c \leq 0$ then*

$$\sup_\Omega u \leq \sup_{\partial\Omega} u^+.$$

Proof. See [23], Theorem 8.1. □

Theorem 3.21 (Interior Hölder continuity). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $f^i \in L^q(\Omega)$ for some $q > n$. Let $u \in W^{1,2}(\Omega)$ be a weak solutionf of $D_i(a^{ij}D_j u) = D_i f^i$ in Ω. Then*

$$\|u\|_{C^{0,\alpha}(\overline{\Omega'})} \leq c(n,q,d,\Lambda_{\max}/\lambda_{\min})\Big(\|u\|_{L^2(\Omega)} + \lambda_{\min}^{-1}\|f\|_{L^q(\Omega)}\Big)$$

where $\Omega' \subset\subset \Omega$, $d = \operatorname{dist}(\Omega', \partial\Omega)$ and $\alpha = \alpha(n, \Lambda_{\max}/\lambda_{\min})$.

Proof. See [23], Theorem 8.24. □

Theorem 3.22 (Boundary Hölder continuity). *Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying a uniform exterior cone condition with cones V on a boundary portion T. Let $f^i \in L^q(\Omega)$ for some $q > n$. Let $u \in W^{1,2}(\Omega)$ be a weak solution of $D_i(a^{ij}D_j u) = D_i f^i$ in Ω. If there exits $K, \alpha_0 > 0$ such that*

$$\operatorname*{osc}_{\partial\Omega \cap B_R(x_0)} u \leq K R^{\alpha_0} \qquad \forall\ x_0 \in T, R > 0.$$

Then

$$\|u\|_{C^{0,\alpha}(\Omega')} \leq c(n,q,d,\alpha_0,V,\Lambda_{\max}/\lambda_{\min})\Big(\|u\|_{C^0(\Omega)} + K + \lambda_{\min}^{-1}\|f\|_{L^q(\Omega)}\Big).$$

Here $\Omega' \subset\subset \Omega \cup T$, $d = \operatorname{dist}(\Omega', \partial\Omega \setminus T)$ and $\alpha = \alpha(n,q,\alpha_0,V,\Lambda_{\max}/\lambda_{\min})$.

Proof. See [23], Theorem 8.29. □

For the sake of completeness let us also mention the existence result in the weak setting.

Theorem 3.23 (Weak existence). *Suppose that $\Omega \subset \mathbb{R}^n$ satisfies an exterior cone condition on all points of $\partial\Omega$. Let $\phi \in C^0(\partial\Omega)$ and $f^i \in L^q(\Omega)$ for some $q > n$.Then there exists a unique weak solution $u \in W^{1,2}_{loc}(\Omega)$ of $D_i(a^{ij}D_j u) = D_i f^i$ satisfying $u = \phi$ on $\partial\Omega$.*

Proof. See [23], Theorem 8.30. □

3.3. Parabolic PDEs in Hölder spaces

Definition 3.24 (Parabolic Hölder spaces). Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $T > 0$. We set $Q_T := \Omega \times (0,T)$, $S_T := \partial\Omega \times (0,T)$ and start with a definition of the parabolic analogue of the spaces $C^k(\overline{\Omega})$:

$$C^{k;\lfloor\frac{k}{2}\rfloor}(\overline{Q_T}) := \left\{ f : \overline{Q_T} \to \mathbb{R} \;\Big|\; \|u\|_{C^{k;\lfloor\frac{k}{2}\rfloor}(\overline{Q_T})} := \sum_{i=0}^{k} \sum_{|\gamma_x|+2|\gamma_t|=i} \sup_{Q_T} |D_t^{\gamma_t} D_x^{\gamma_x} f| < \infty \right\}$$

Let us denote the Hölder coefficients of a function $f : Q_T \to \mathbb{R}$ by

$$[f]_{x,\alpha,Q_T} := \sup_{\substack{(x,t),(y,t)\in Q_T \\ x\neq y}} \frac{|f(y,t) - f(x,t)|}{|y-x|^\alpha}$$

and

$$[f]_{t,\alpha,Q_T} := \sup_{\substack{(x,s),(x,t)\in Q_T\\ s\neq t}} \frac{|f(x,t)-f(x,s)|}{|t-s|^\alpha}.$$

The parabolic Hölder spaces are defined as

$$C^{k,\alpha;\lfloor\frac{k}{2}\rfloor,\frac{\alpha}{2}}(\overline{Q_T}) := \left\{ u \in C^{k,\lfloor\frac{k}{2}\rfloor}(\overline{Q_T}) \;\middle|\; \|u\|_{C^{k,\alpha;\lfloor\frac{k}{2}\rfloor,\frac{\alpha}{2}}(\overline{Q_T})} < \infty \right\}$$

with

$$\|u\|_{C^{k,\alpha;\lfloor\frac{k}{2}\rfloor,\frac{\alpha}{2}}(\overline{Q_T})} := \|u\|_{C^{k;\lfloor\frac{k}{2}\rfloor}(\overline{Q_T})} + \sum_{2|\gamma_t|+|\gamma_x|=k} [D_t^{\gamma_t} D_x^{\gamma_x} u]_{x,\alpha,Q_T} + \sum_{0<k+\alpha-2|\gamma_t|-|\gamma_x|<2} [D_t^{\gamma_t} D_x^{\gamma_x} u]_{t,\beta,Q_T}$$

where $2\beta := k+\alpha-2|\gamma_t|-|\gamma_x|$.

Exercise I.8. What are the components of the $\|\cdot\|_{C^{2,\alpha;1,\frac{\alpha}{2}}(\overline{Q_T})}$-norm?

Definition 3.25 (Linear uniformly parabolic differential operators). Let $a^{ij}, b^k, c \in C^0(\overline{Q_T})$ and suppose that $[a^{ij}]$ is symmetric. Let $u \in C^{2,1}(\overline{Q_T})$. The operator $\partial/\partial t - L$ defined by

$$\frac{\partial u}{\partial t} - Lu := \frac{\partial u}{\partial t} - a^{ij} D_{ij} u + b^k D_k u + cu$$

is called parabolic. It is called uniformly parabolic in Q_T if additionally

$$0 < \lambda_{\min} \leq a^{ij}(x,t)\xi_i\xi_j \leq \Lambda_{\max} < \infty$$

holds for all $(x,t) \in \overline{Q_T}$ and all $\xi \in$ \$.

Theorem 3.26 (Existence for the parabolic Dirichlet boundary value problem). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{2,\alpha}$-boundary and $T > 0$. Let $\partial/\partial t - L$ be uniformly parabolic in Q_T. Suppose that $a^{ij}, b^k, c, f \in C^{0,\alpha,0,\frac{\alpha}{2}}(\overline{Q_T})$, $\phi \in C^{2,\alpha,1,\frac{\alpha}{2}}(\overline{S_T})$ and $u_0 \in C^{2,\alpha}(\overline{\Omega})$. If the compatibility conditions*

$$\phi(\cdot,0) = u_0, \quad \frac{\partial}{\partial t}\Big|_{t=0}\phi = a^{ij}(\cdot,0)D_{ij}u_0 - b^k(\cdot,0)D_k u_0 - c(\cdot,0)u_0 + f(\cdot,0)$$

are satisfied on $\partial\Omega$. Then the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - Lu = f & \text{in } \Omega \times (0,T) \\ u = \phi & \text{on } \partial\Omega \times (0,T) \\ u(\cdot,0) = u_0 & \text{on } \Omega \times \{0\} \end{cases}$$

has a unique solution $u \in C^{2,\alpha,1,\frac{\alpha}{2}}(\overline{Q_T})$ which satisfies

$$\|u\|_{C^{2,\alpha;1,\frac{\alpha}{2}}(\overline{Q_T})} \leq C\Big(\|f\|_{C^{0,\alpha;0,\frac{\alpha}{2}}(\overline{Q_T})} + \|\phi\|_{C^{2,\alpha;1,\frac{\alpha}{2}}(\overline{S_T})} + \|u_0\|_{C^{2,\alpha}(\overline{\Omega})} \Big).$$

If the coefficients and right hand sides are more regular the solution will be more regular too. However, to obtain more regular solutions up to $t = 0$ one also has to impose higher order compatibility conditions.

Proof. See [38], Chapter 5, Theorem 5.2. □

Theorem 3.27 (Existence for the parabolic Neumann boundary value problem). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{2,\alpha}$-boundary and $T > 0$. Let $\partial/\partial t - L$ be uniformly parabolic in Q_T. Suppose that $a^{ij}, b^k, c, f \in C^{0,\alpha,0,\frac{\alpha}{2}}(\overline{Q_T})$, $\beta^k, \gamma, \phi \in C^{1,\alpha;0,\frac{\alpha}{2}}(\overline{S_T})$ and $u_0 \in C^{2,\alpha}(\overline{\Omega})$. If the compatibility condition*

$$\phi(\cdot,0) = \beta^k(\cdot,0)D_k u_0 + \gamma(\cdot,0)u_0 \quad \text{on } \partial\Omega$$

is satisfied and the transversality condition holds, i.e. for the outward unit normal μ of $\partial\Omega \times (0,T)$ we have

$$\langle \beta, \mu \rangle > 0 \quad \text{on } \partial\Omega \times (0,T).$$

Then the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - Lu = f & \text{in } \Omega \times (0,T) \\ \beta^k D_k u + \gamma u = \phi & \text{on } \partial\Omega \times (0,T) \\ u(\cdot,0) = u_0 & \text{on } \Omega \times \{0\} \end{cases} \tag{3.4}$$

has a unique solution $u \in C^{2,\alpha,1,\frac{\alpha}{2}}(\overline{Q_T})$ which satisfies

$$\|u\|_{C^{2,\alpha;1,\frac{\alpha}{2}}(\overline{Q_T})} \le C\Big(\|f\|_{C^{0,\alpha;0,\frac{\alpha}{2}}(\overline{Q_T})} + \|\phi\|_{C^{1,\alpha;0,\frac{\alpha}{2}}(\overline{S_T})} + \|u_0\|_{C^{2,\alpha}(\overline{\Omega})}\Big).$$

If the coefficients and right hand sides are more regular the solution will be more regular too. However, to obtain more regular solutions up to $t = 0$ one also has to impose higher order compatibility conditions.

Proof. See [38], Chapter 5, Theorem 5.3. □

Remark 3.28 (Differentiable functions defined on hypersurfaces/ boundaries). As in the differential geometry section we say that a function is in $C^k(S)$ for some C^k-hypersurface $S \subset \mathbb{R}^n$ if the composition with a chart has that regularity as a map from an open subset of $\mathbb{R}^n$ into $\mathbb{R}$. Let $k \ge 0$ and a domain $\Omega \subset \mathbb{R}^n$ with C^k-boundary. If $\phi \in C^k(\overline{\Omega})$ then $\phi|_{\partial\Omega}$ defines a function in $C^k(\partial\Omega)$. Conversely, if $\phi \in C^k(\partial\Omega)$ then there exists $\overline{\phi} \in C^k(\overline{\Omega})$ such that both functions agree on the boundary and have equivalent norms. The same result carries over to the parabolic setting and to Hölder norms where the norms are compute in charts and the global norm is defined via a partition of unity. That is how we understand terms like $\|\phi\|_{C^{2,\alpha;1,\frac{\alpha}{2}}(S_T)}$.

Remark 3.29 (PDEs on manifolds). In the situation where $\Omega \subset \mathbb{R}^n$ is replaced by a smooth Riemannian manifold (M^n, g) we have to modify the definition of the norms. They can defined locally via charts and globally via a partition of unity. Note that in this context the norms depend on the choice of atlas. Once this is done, one obtains the same existence results for the linear Dirichlet and Neumann problem as above. Furthermore, one could allow a time dependent metric $g(\cdot,t)$.

The most important tool for second order parabolic equations is the maximum principle. Before we mention it we define sub- and supersolutions.

Definition 3.30 (Super- and subsolutions). Let $v^+, v^- \in C^{2,1}(\Omega \times (0,T)) \cap C^0(\Omega \times [0,T])$. We say that v^+ is a supersolutions of (3.4) if it satisfies

$$\begin{cases} \dfrac{\partial v^+}{\partial t} - Lv^+ \ \geq \ f_1 & \text{in } \ \Omega \times (0,T) \\ Nv^+ := \beta^k D_k v^+ + \gamma v^+ \ \geq \ f_2 & \text{on } \ \partial\Omega \times (0,T) \\ v^+ \ \geq \ u_0 & \text{on } \ \Omega \times \{0\}. \end{cases}$$

The function v^- is called subsolution if the opposite inequalities hold.

Now we can state the version of the maximum principles which we use in this work.

Theorem 3.31 (Parabolic maximum principle). *Let $u \in C^0(\overline{\Omega} \times [0,T]) \cap C^{2,1}(\overline{\Omega} \times (0,T))$ be a solution of* (3.4)*. Assume that L and N have bounded coefficients, that $\partial/\partial t - L$ is uniformly parabolic and that the Neumann condition is oblique. If v^+ and v^- are super- and subsolutions of* (3.4) *then $v^- \leq u \leq v^+$ in $\overline{Q}_T$.*

Proof. Note that for $w := v^+ - u$ and $w := u - v^-$ we have $\partial w/\partial t - Lw \geq 0$, $Nw \geq 0$ and $w(\,.\,,0) \geq 0$. So we can reduce the proof to the case of the upper bound for $f_1 = 0, f_2 = 0$ and $u_0 = 0$. This proof is contained in [53] Chapter 3, Section 3, Theorem 5,6 and 7. Furthermore Stahl proved in [57] the generalization which in particular allows for the more general operator N which occurs here. ☐

Corollary 3.32. *If $f_1 \equiv 0$ and $f_2 \equiv 0$, then $v^+ := \max_\Omega u_0$ is a supersolution if*

$$c \max_\Omega u_0 \geq 0 \qquad \text{and} \qquad \gamma \max_\Omega u_0 \geq 0.$$

Furthermore $v^- := \min_\Omega u_0$ is a subsolution if

$$c \min_\Omega u_0 \leq 0 \qquad \text{and} \qquad \gamma \min_\Omega u_0 \leq 0.$$

Obviously, these inequalities are all satisfied for $c \equiv 0$ and $\gamma \equiv 0$.

Corollary 3.33 (Comparison to the ODE). *Assume that $f_1 \equiv 0$, $f_2 \equiv 0$, $\gamma = 0$ and $c(x,t) = c(t)$. Then v^+ given as a solution of*

$$\text{(ODE)} \begin{cases} \dfrac{\partial v^+}{\partial t} + cv^+ \geq 0 & \text{on } \Omega \times (0,T) \\ v^+(0) = \max_\Omega u_0 \end{cases}$$

is a supersolution. Furthermore, the function v^- satisfying the same ODE *with the reverse inequality and the initial value* $\min_\Omega u_0$ *is a subsolution.*

Part II.

Nonlinear elliptic PDEs of second order

4. General theory for quasilinear problems

4.1. Fixed point theorems: From Brouwer to Leray-Schauder

We start by recalling Brouwer's fixed point theorem.

Theorem 4.1 (Brouwer). *Let us denote by $\overline{B}$ the closed ball $\overline{B}_r(x_0) \subset \mathbb{R}^n$ of radius r centered at x_0. If $f : \overline{B} \to \overline{B}$ is continuous then f has a fixed point.*

Idea of proof. Suppose that $f : \overline{B}_r(x_0) \to \overline{B}_r(x_0)$ has no fixed point. Then x and $f(x)$ always span a line. Therefore, we can define a map

$$R : \overline{B}_r(x_0) \to \partial\overline{B}_r(x_0) : x \mapsto Rx$$

where Rx is given as the point of intersection between the line segment starting from $f(x)$ in direction x and $\partial\overline{B}_r(x_0)$. Note that R is a retraction, i.e. R is continuous and satisfies $Rx = x$ for all x in $\partial\overline{B}_r(x_0)$. If we regard the set of points in $\overline{B}_r(x_0)$ as a membrane. Then the map R describes a continuous deformation of such a membrane which moves all points to the boundary. Intuitively, the membrane will be torn apart.

For a proof of the non-existence of such a retraction we refer to literature, e.g. [67], Section 1.14 or [55], Section 1.2. The proof also occurred as an exercise on homework sheet four in Michael Eichmair's class Differential Geometry II from last semester. □

This result can be extended to compact, convex subsets of a Banach space.

Theorem 4.2 (Schauder). *Let $(X, \|\cdot\|)$ be a Banach space and $A \subset X$ compact and convex. If $T : A \to A$ is continuous then T has a fixed point.*

Proof. Since A is compact for every $k \in \mathbb{N}$ we find a finite number $N \in \mathbb{N}$ of points $x_i \in A$ such that

$$A \subset \bigcup_{i=1}^{N} B_{1/k}(x_i), \qquad N = N(k), \quad x_i = x_i(k).$$

Let us denote the convex hull of $\{x_i \mid 1 \leq k \leq N\}$ by A_k^{co}. Note that $A_k^{co} \subset A$. We define the continuous map

$$J_k : A \to A_k^{co} : x \mapsto J_k(x) := \frac{\sum_{i=1}^{N} \operatorname{dist}\left(x, A \setminus B_{1/k}(x_i)\right) x_i}{\sum_{i=1}^{N} \operatorname{dist}\left(x, A \setminus B_{1/k}(x_i)\right)}.$$

Since A_k^{co} is convex and generated by a finite number of elements it is homeomorphic via a map h to a closed ball $\overline{B}$ in some Euclidean space. From Brouwer's Theorem 4.1 we see that $h \circ (J_k \circ T) \circ h^{-1} : \overline{B} \to \overline{B}$ has a fixed point and thus the same holds for $J_k \circ T$ restricted to A_k^{co}.

We denote the sequence of fixed points of $J_k \circ T$ by $(x^{(k)})_{k\in\mathbb{N}}$. Note that A is compact. Therefore, there exists a subsequence $(x^{(k_l)})_{l\in\mathbb{N}}$ which converges to some $x \in A$ as k_l tends to infinity. We observe that

$$\left|Tx - x\right| = \lim_{l\to\infty} \left|Tx^{(k_l)} - x^{(k_l)}\right| = \lim_{l\to\infty} \left|Tx^{(k_l)} - (T \circ J_{k_l})(x^{(k_l)})\right| \leq \lim_{l\to\infty} \frac{1}{k_l} = 0.$$

Thus, x is a fixed point of T. □

Corollary 4.3. *Let $(X, \|\cdot\|)$ be a Banach space and $A \subset X$ closed and convex. If $T : A \to A$ is continuous and TA is relativly compact then T has a fixed point.*

Exercise II.1. Try to prove Corolloary 4.3. Hint: Ist the set $(\overline{TA})^{co}$ compact?

Based on Corollary 4.3 we obtain the version which is important for our application.

Theorem 4.4 (Schaefer). *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ continuous and compact. If*

$$\mathcal{M} = \{x \in X \mid \exists \sigma \in [0,1] : \; x = \sigma T x\}$$

is bounded then T has a fixed point.

Proof. Let $\mathcal{M}$ be strictly bounded by $M > 0$. Define the map $T^* : \overline{B_M(0)} \to \overline{B_M(0)}$ by

$$T^*x \;:=\; \begin{cases} Tx & \text{for} \quad \|Tx\|_X \leq M, \\ \dfrac{M}{\|Tx\|_X} Tx & \text{for} \quad \|Tx\|_X > M. \end{cases}$$

Note that T^* is continuous and $\overline{B_M(0)}$ is closed and convex. Since T is compact $T\overline{B_M(0)}$ is relatively compact. Thus, also $T^*\overline{B_M(0)}$ is relatively compact and by Corollary 4.3 T^* has a fixed point x^*. Now, suppose that $\|Tx^*\|_X > M$. On the one hand

$$x^* \;=\; T^*x^* \;=\; \frac{M}{\|Tx^*\|_X} Tx^*$$

yields $\|x^*\|_X = M$. On the other hand $x^* = \sigma T x^*$ ($\sigma = M/\|Tx^*\|_X$) implies $\|x^*\|_X < M$. Therefore, $\|Tx^*\|_X \leq M$ and $x^* = T^*x^* = Tx^*$. □

One can even allow a more general dependence on the paramenter σ.

Theorem 4.5 (Leray-Schauder). *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \times [0,1] \to X$ continuous and compact. If $T(\cdot, 0) = 0$ and*

$$\mathcal{M} = \{x \in X \mid \exists \sigma \in [0,1] : \; x = T(x, \sigma)\}$$

is bounded then $T(\cdot, 1)$ has a fixed point.

Proof. For our purpose the theorem of Schaefer will be sufficient. Therefore, we skip the proof and refer for it to [23], Theorem 11.6. □

4.2. Reduction to a priori estimates in the $C^{1,\beta}$-norm

We consider the following family of quasilinear Dirichlet problems of second order

$$(\mathrm{DP})_\sigma \begin{cases} Q_\sigma u := a^{ij}(\cdot,u,Du)D_{ij}u + \sigma b(\cdot,u,Du) = 0 & \text{in } \Omega, \\ u = \sigma\phi & \text{on } \partial\Omega \end{cases}$$

with $\sigma \in [0,1]$, continuous coefficients $a^{ij}, b \in C^0(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ and symmetric matrix $[a^{ij}]$. Furthermore, we set $Q := Q_1$ and $(\mathrm{DP}) := (\mathrm{DP})_1$.

Definition 4.6. Let $A \subset \mathbb{R}^n$. The operator Q is called elliptic in A if $[a^{ij}(x,z,p)]$ is positive definite for all $(x,z,p) \in A \times \mathbb{R} \times \mathbb{R}^n$. Furthermore, Q is called elliptic w.r.t. $v \in C^1(A)$ if $[a^{ij}(x,v(x),Dv(x))]$ is positive definite for all $x \in A$.

Let T be the operator which assignes to v the solution u of the linear problem

$$a^{ij}(\cdot,v,Dv)D_{ij}u + b(\cdot,v,Dv) = 0 \textit{ in } \Omega, \qquad u = \phi \text{ on } \partial\Omega. \tag{4.1}$$

We see that the existence of a fixed point of T guarantees the existence of a solution of (DP). Based on this observation Theorem 4.4 yields a first criterion for existence.

Theorem 4.7 (Existence criterion: $C^{1,\beta}$-version). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{2,\alpha}$-boundary. Let Q be elliptic in $\overline{\Omega}$ with coefficients $a^{ij}, b \in C^{0,\alpha}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ and $\phi \in C^{2,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$. If there exists some $\beta \in (0,1)$ such that the set*

$$\{u \in C^{2,\alpha}(\overline{\Omega}) \mid \exists\, \sigma \in [0,1] \;:\; u \textit{ solves } (\mathrm{DP})_\sigma\}$$

is bounded in $C^{1,\beta}(\overline{\Omega})$ independently of σ. Then (DP) has a solution in $C^{2,\alpha}(\overline{\Omega})$.

Proof. Let $v \in C^{1,\alpha}(\overline{\Omega})$. Then (4.1) is a linear, uniformly elliptic problem for u with coefficients in $C^{0,\alpha\beta}(\overline{\Omega})$. By Theorem 3.6 there exists a unique solution $u \in C^{2,\alpha\beta}(\overline{\Omega}) \subset C^{1,\beta}(\overline{\Omega})$. Thus, the operator

$$T \;:\; C^{1,\beta}(\overline{\Omega}) \to C^{1,\beta}(\overline{\Omega}) \;:\; v \mapsto Tv := u,$$

is well defined. We need to show that T has a fixed point. By Theorem 4.4 this follows if T is a continuous, compact operator and the set $\mathcal{M} = \{u \in C^{1,\beta}(\overline{\Omega}) \mid \exists \sigma \in [0,1] : u = \sigma T u\}$ is bounded. The latter is true by assumption[1] so it remains to verify continuity and compactness.

Compactness: By Theorem 3.6 we have

$$\|Tv\|_{C^{2,\alpha\beta}(\overline{\Omega})} \le c\left(\|Tv\|_{C^0(\overline{\Omega})} + \|\phi\|_{C^{2,\alpha\beta}(\overline{\Omega})} + \||b(\cdot,v,Dv)\|_{C^{0,\alpha\beta}(\overline{\Omega})}\right)$$

with $c = c\left(n,\alpha,\Omega,\|a^{ij}(\cdot,v,Dv)\|_{C^{0,\alpha\beta}(\overline{\Omega})}, \min_\Omega \lambda(\cdot,v,Dv)\right)$ and from Theorem 3.2 we obtain the C^0 estimate

$$\|Tv\|_{C^0(\overline{\Omega})} \le \|\phi\|_{C^0(\overline{\Omega})} + c(\operatorname{diam}\Omega)\left\|\frac{b(\cdot,v,Dv)}{\lambda(\cdot,v,Dv)}\right\|_{C^0(\overline{\Omega})}.$$

[1]Note that $v \in \mathcal{M} \Rightarrow v = \sigma T v \in C^{2,\alpha\beta}(\overline{\Omega}) \Rightarrow v = \sigma T v \in C^{2,\alpha}(\overline{\Omega})$.

Therefore, we see that T maps bounded subsets of $C^{1,\beta}(\overline{\Omega})$ into bounded subsets of $C^{2,\alpha\beta}(\overline{\Omega})$ which by Arzelà-Ascoli are relatively compact in $C^{1,\beta}(\overline{\Omega})$, i.e. T is compact.

Continuity: Suppose that $\{v_m\}_{m\in\mathbb{N}} \subset C^{1,\beta}(\overline{\Omega})$ converges in the $C^{1,\beta}$-norm to some v. In particular this sequence is bounded in the $C^{1,\beta}$-norm. As above this implies that $\{Tv_m\}_{m\in\mathbb{N}}$ is bounded in the $C^{2,\alpha\beta}$-norm. Thus by Arzelà-Ascoli $\{Tv_m\}_{m\in\mathbb{N}}$ is relatively compact in $C^2(\overline{\Omega})$. This is equivalent to the existence of a subsequence $\{Tv_{m_k}\}_{k\in\mathbb{N}}$ which converges in the $C^2(\overline{\Omega})$-norm to some $u \in C^2(\overline{\Omega})$. Since a^{ij} and b are continuous this yields

$$0 = \lim_{k\to\infty} \Big(a^{ij}(x, v_{m_k}(x), Dv_{m_k}(x)) D_{ij}(Tv_{m_k}(x)) \;+\; b(x, v_{m_k}(x), Dv_{m_k}(x)) \Big)$$

$$= a^{ij}(x, v(x), Dv(x)) D_{ij}u(x) \;+\; b(x, v(x), Dv(x)) \qquad \forall\, x \in \Omega.$$

This shows that $Tv = u$. Finally, all subsequences have to converge to the same limit which shows that T is continuous. □

4.3. Reduction to a priori estimates in the C^1-norm

The previous result shows that the existence proof is reduced to a priori estimates in $C^{1,\beta}(\overline{\Omega})$. It turns out that the Hölder estimate for Du can be carried out under very mild assumptions on the operator Q. This will help us to then formulate an existence criterion based on C^1 a priori estimates.

Theorem 4.8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2-boundary. Let Q be elliptic in $\overline{\Omega}$ with coefficients $a^{ij} \in C^1(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$, $b \in C^0(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ and let $\phi \in C^2(\overline{\Omega})$. If $u \in C^2(\overline{\Omega})$ solves* (DP) *then there exists $\beta \in (0,1)$ such that*

$$[Du]_{\beta,\Omega} \le C \left(n,\ \Omega,\ K,\ \min\nolimits_{U_K} \lambda,\ \|a^{ij}\|_{C^1(U_K)},\ \|b\|_{C^0(U_K)},\ \|\phi\|_{C^2(\overline{\Omega})}\right) < \infty$$

with $K := \|u\|_{C^1(\overline{\Omega})}$ and $U_K := \overline{\Omega} \times [-K,K] \times [-K,K]^n$.

Remark 4.9. The result was first obtained independently by De Giorgi and Nash for linear operators in divergence form: $Lu = D_i(a^{ij}(x)D_ju)$. It was a major break through in the study of nonlinear elliptic PDEs. Later Morrey and Stampachia extended the work to linear elliptic operators of general form. The theorem as it is stated above goes back to Ladyžhenskaya and Ural'ceva.

Proof. The proof can be found in [23], Theorem 13.7. For our later application to the prescribed mean curvature equation it will be enough to consider operators in divergence form, i.e. $a \in C^1(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, $b \in C^0(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R})$:

$$\operatorname{div} a(\cdot, u, Du) + b(\cdot, u, Du) = 0. \tag{4.2}$$

In that case the proof is contained in [23], Theorem 13.1 (interior estimate) and Theorem 13.2 (boundary estimate). We will only give a sketch of these arguments here.

Interior estimate: Multiplying (4.2) by a test function $\xi \in C_0^1(\Omega)$, integrating over Ω and doing an integration by parts on the first term yields

$$\int_\Omega \Big[a^i(\cdot,u,Du)D_i\xi - b(\cdot,u,Du)\xi\Big]\,dx = 0.$$

We put $\xi := D_k\eta$ and perform an integration by parts on the first term w.r.t. D_k:

$$\int_\Omega \left[\left(\frac{\partial a^i}{\partial x^k} + \frac{\partial a^i}{\partial z}D_k u + \frac{\partial a^i}{\partial p^j}D_{kj}u\right)D_i\eta + bD_k\eta\right]dx = 0$$

where all partial derivatives of a^i and b are evaluated at $(x, u(x), Du(x))$. Thus we get

$$\int_\Omega (a^{ij}(x)D_j w + f^i)D_i\eta\,dx = 0$$

with $w := D_k u$ and

$$a^{ij}(x) := \frac{\partial a^i}{\partial p^j}(x,u(x),Du(x)),$$

$$f^i(x) := \left(\frac{\partial a^i}{\partial x^k} + p^k\frac{\partial a^i}{\partial z} + \delta^i_k b\right)(x,u(x),Du(x)).$$

So w is a weak solution of the linear, uniformly elliptic equation $D_i(a^{ij}(x)D_jw) = -D_if^i$ with bounded coefficients and Theorem 3.21 yields the interior estimate.

Boundary estimate: W.l.o.g. we may assume $\phi \equiv 0$ (or consider $u-\phi$ instead of u). Let $x_0 \in \partial\Omega$. Choose a small ball $B(x_0)$ and a C^2-diffeomorphism ψ which flattens out the boundary locally near x_0, i.e. such that

$$D^+ := \psi(B(x_0)\cap\Omega) \subset \mathbb{R}^{n-1}\times[0,\infty), \qquad \partial_0D^+ := \psi(B(x_0)\cap\partial\Omega) \subset \mathbb{R}^{n-1}\times\{0\}.$$

In the new coordinates $y = \psi(x)$ the function $v := (u\circ\psi^{-1})$ satisfies again an equation of the form

$$\operatorname{div}\tilde a(\cdot,v,Dv) + \tilde b(\cdot,v,Dv) = 0 \tag{4.3}$$

which implies that $w := D_{y^k}v$ is a weak solution of a linear equation $D_i(\tilde a^{ij}D_jw) = -D_i\tilde f^i$ where the derivatives are taken w.r.t. y now. Let $1 \le k \le n-1$: Since we assumed that $u = 0$ on $\partial\Omega$ we have $u\circ\psi^{-1} = 0$ on ∂_0D^+ and thus $w = 0$ on ∂_0D^+. By Theorem 3.22 this implies a Hölder estimate for w in $D'\cap D^+$ for any $D' \subset\subset D$.

Note that we can not apply this strategy to get the estimate for $w = D_nv$ since we have no information about D_nv on the boundary. Instead we try to verify that

$$\int_{B_R(y_0)\cap D^+} |Dw|^2dy \le cR^{n-2+2\alpha}, \qquad w = D_nv \tag{4.4}$$

for $y_0 \in D'\cap D^+$ and $R > 0$ sufficiently small such that $B_{2R}(y_0) \subset D$. Using the Hölder inequality $\|u\|_{L^1(B_R)} \le cR^{n/2}\|u\|_{L^2(B_R)}$ and the Morrey estimate, Theorem 3.16, we will then obtain the desired Hölder estimate. In order to prove (4.4) we solve equation (4.3) for $D_{nn}v$: $D_{nn}v = C^{jk}D_{jk}v + C$ with $1 \le k \le n-1$. This shows that it suffices to verify (4.4) for $w = D_k v$ and $1 \le k \le n-1$. So we put $w = D_k v$ and choose $\eta = \rho^2(w-c)$ with $c = w(y_0)$ if $B_{2R}(y_0) \subset D^+$ and zero otherwise. Plugging this into (4.3) yields

$$0 = \int_{D^+} \Big[\rho^2 \tilde{a}^{ij} D_i w D_j w + 2\rho\tilde{a}^{ij} D_i\rho D_j w(w-c) + \rho^2 \tilde{f}^i D_i w + 2\rho(w-c)\tilde{f}^i D_i\rho\Big]\, dx.$$

Now we choose $\rho \in C_c^1(B_{2R}(y);[0,1])$ such that $\rho \equiv 1$ on $B_r(y)$. Furthermore, we use the ellipticity of $[a^{ij}]$ and Young's inequality $ab \le \varepsilon a^2/2 + b^2/2\varepsilon$ to compute

$$\begin{aligned}
\lambda \int_{B_R(y_0)\cap D^+} |Dw|^2\, dx &\le \int_{B_{2R}(y_0)} \rho^2 \tilde{a}^{ij} D_i w D_j w\, dy \\
&\le \int_{B_{2R}(y_0)\cap D^+} \Big(c_1|D\rho||Dw||w-c| + c_2|Dw| + c_3|w-c||D\rho|\Big)\, dy \\
&\le c_4 \int_{B_{2R}(y_0)\cap D^+} \Big(1 + |D\rho|^2|w-c|^2 + (\varepsilon_1^2+\varepsilon_2^2)|Dw|^2\Big)\, dy.
\end{aligned}$$

Choosing ε_1 and ε_2 small enough, e.g. $(\varepsilon_1^2+\varepsilon_2^2) \le \lambda/2$, we can absorb the last term into the right hand side. Using the Hölder continuity of w and choosing ρ such that $|D\rho| \le 2/R$ we obtain

$$\int_{B_R(y_0)\cap D^+} |Dw|^2 \le c_5\left(R^n + R^{n-2}\sup_{B_{2R}(y_0)\cap D^+}|w-c|^2\right) \le c_6 R^{n-2+2\alpha}.$$

This yields the Morrey estimate for $w = D_k v$ and as explained above also for $D_n v$. Finally, we rewrite the estimate in terms of u and repeat it in a finite number of balls which cover $\partial\Omega$. This yields the desired result. □

Together with Theorem 4.7 this improves our existence criterion.

Corollary 4.10 (Existence criterion: C^1-version). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{2,\alpha}$-boundary. Let Q be elliptic in $\overline{\Omega}$ with coefficients $a^{ij} \in C^1(\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^n)$, $b \in C^{0,\alpha}(\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^n)$ and $\phi \in C^{2,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$. If the set*

$$\{u \in C^{2,\alpha}(\overline{\Omega}) \mid \exists\, \sigma \in [0,1] \ :\ u \text{ solves } (\mathrm{DP})_\sigma\}$$

is bounded in $C^1(\overline{\Omega})$ independently of σ. Then (DP) *has a solution in $C^{2,\alpha}(\overline{\Omega})$.*

5. The prescribed mean curvature problem

In the following we are interested in Dirichlet problem for the prescribed mean curvature equation (PMC) := $(\mathrm{PMC})_1$ where

$$(\mathrm{PMC})_\sigma \begin{cases} \operatorname{div}\left(\dfrac{Du}{\sqrt{1+|Du|^2}}\right) - \sigma H(\cdot, u, Du) = 0 & \text{in } \Omega \\ u = \sigma\phi & \text{on } \partial\Omega \end{cases}$$

for a given function H.

5.1. C^0-estimate

Let us first derive a comparison principle for quasilinear equations. It will be useful for the proof of the C^0-estimate as well as for the proof of the boundary gradient estimate.

Theorem 5.1 (**Comparison principle for quasilinear elliptic PDEs**). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $u, v \in C^2(\overline{\Omega})$. Suppose that Q is a quasilinear operator with continuous coefficients such that*

(a) Q is uniformely elliptic with respect to u or v,

(b) a^{ij} is independent of z,

(c) b is non-increasing in z,

(d) a^{ij} and b are continuously differentiable in p.

If $Qu \geq Qv$ in Ω and $u \leq v$ on $\partial\Omega$ then $u \leq v$ in $\overline{\Omega}$. If $Qu > Qv$ in Ω then the weaker assumptions $(a) - (c)$ yield the stronger conclusion $u < v$ in Ω.

Proof. Let Q be elliptic w.r.t. u. We put $w := u - v$ and $u_s := su + (1-s)v$. Using $(b), (c)$ and (d) we compute on the set where $w > 0$:

$$\begin{aligned} 0 &\leq Qu - Qv \\ &= a^{ij}(\cdot, Du)D_{ij}(u-v) + \left[a^{ij}(\cdot, Du) - a^{ij}(\cdot, Dv)\right]D_{ij}v \\ &\quad + \left[b(\cdot, u, Du) - b(\cdot, u, Dv)\right] + \left[b(\cdot, u, Dv) - b(\cdot, v, Dv)\right] \\ &\leq a^{ij}(\cdot, Du)D_{ij}(u-v) + \left[\left(\int_0^1 \frac{\partial a^{ij}}{\partial p^k}\Big|_{(\cdot, Du_s)} ds D_{ij}v\right) + \left(\int_0^1 \frac{\partial b}{\partial p^k}\Big|_{(\cdot, u, Du_s)} ds\right)\right] D_k w \\ &=: a^{ij}D_{ij}w + b^k D_k w =: Lw. \end{aligned}$$

We see that $Lw \geq 0$ in $\{w > 0\}$ and $w \leq 0$ on $\partial\Omega$. Therefore, $w = 0$ on $\partial\{w > 0\}$ and we can use (a) and apply the linear maximum principle, Theorem 3.2 to conclude that

$w \le 0$ in $\{w > 0\}$. Thus $w = u - v \le 0$ in $\overline{\Omega}$. If we have a strict inequality, at a critical point we have $Du = Dv$ and thus

$$0 < a^{ij} D_{ij} w$$

even without computing the partial derivatives w.r.t. p^k. Therefore, w can not have a nonnegative maximum in Ω, i.e. $w = u - v < 0$ in Ω. Finally, if Q is uniformly elliptic w.r.t v then we obtain the reverse inequalities and apply the minimum principle. □

Remark 5.2. The Theorem can be relaxed to allow $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$. In this situation the uniform ellipticity in (a) is replace by locally uniform ellipticity. This is important if one tries to derive existence results for C^0 boundary data.

This works because the corresponding linear result, Theorem 3.2 does not actually require uniform ellipticity but only ellipticity together with demanding $b^k/[a^{kk}]$ to be locally bounded. Furthermore, it can be extended to hold for $u \in C^1(\Omega) \cap W^{2,n}(\Omega) \cap C^0(\overline{\Omega})$.

Corollary 5.3 (Uniqueness). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $\phi \in C^0(\overline{\Omega})$. Suppose that the operator Q satisfies the conditions of Theorem 5.1. Then there is at most one solution $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ satisfying $Qu = 0$ in Ω and $u = \phi$ on $\partial\Omega$.*

Proof. Clear. □

Proposition 5.4. *Let $\Omega \subset \mathbb{R}^n$. Let $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ be a solution of* $(\mathrm{PMC})_\sigma$. *If*

$$\sup_{\Omega \times \mathbb{R} \times \mathbb{R}^n} |H| \le \frac{n}{R_\Omega}, \qquad R_\Omega := \frac{\operatorname{diam} \Omega}{2}$$

and H is non-decreasing in z. Then the following estimate holds

$$\sup_\Omega |u| \le \sup_\Omega |\phi| + R_\Omega.$$

Proof. Exercise. Hint: Use the comparison principle Theorem 5.1 and compare the solution to spherical caps, i.e. the graphs of the functions

$$v^\pm : \overline{B}_{R_\Omega}(x_0) \to \mathbb{R} : x \mapsto v^\pm(x) := \pm \left(\sup_\Omega |\phi| + \sqrt{R_\Omega^2 - |x - x_0|^2} \right)$$

for some $x_0 \in \mathbb{R}^n$ such that $\Omega \subseteq B_{R_\Omega}(x_0)$. □

The following theorem provides another tool to prove a priori estimates for general quasilinear elliptic operators:

Theorem 5.5. *Let Q be a quasilinear operator which is elliptic in $\overline{\Omega} \subset \mathbb{R}^n$. Let $g \in L^n_{loc}(\mathbb{R}^n)$ and $h \in L^n(\Omega)$ be non-negative functions which satisfy*

$$\int_\Omega h^n \le \int_{\mathbb{R}^n} g^n.$$

Let $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ be a solution of $Qu := a^{ij}(\cdot, u, Du) D_{ij} u + b(\cdot, u, Du) = 0$. If

$$\frac{b(x,z,p) sgn(z)}{n(\det[a^{ij}(x,z,p)])^{1/n}} \le \frac{h(x)}{g(p)} \qquad \forall\ (x,z,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$$

then u satisfies the estimate

$$\sup_\Omega |u| \le \sup_{\partial\Omega} |u| + c(g,h) \operatorname{diam}(\Omega).$$

Proof. A proof can be found in [23], Theorem 10.5. □

Exercise II.2. Use Theorem 5.5 to prove C^0-a priori estimates for solutions of (PMC). Which condition do you have to impose on the right hand side, i.e. on H?

Remark 5.6. In general, Theorems which provide estimates for a broad class of (non-linear) equations will not provide optimal results. However, they are useful to get a first intuition for the conditions which might be needed. Better results can be expected from methods which make use of the special structure of the equation. Following this strategy we skip the proof of Theorem 5.5 and focus instead on a C^0-a priori estimate which incoorporates the special structure of the (PMC) problem.

First, we will prove an estimate in $L^{\frac{n}{n-1}}$ which we wil then turn into a C^0-estimate using Stampacchia's lemma.

Proposition 5.7. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^1-boundary and $\phi \in C^0(\partial\Omega)$. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a solution of $(\mathrm{PMC})_\sigma$ for some $\sigma \in [0,1]$ and $H \in C^{0,1}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ such that H is increasing in z. If there exists $\varepsilon \in (0,1]$ such that*

$$\left| \int_\Omega H(\cdot, 0, D\eta)\eta \,\mathrm{d}x \right| \le (1-\varepsilon) \int_\Omega |D\eta| \,\mathrm{d}x \qquad \forall\ \eta \in C_0^1(\Omega) \tag{5.1}$$

Then $\hat{u}_k := \max\{u - k, 0\}$ and $\check{u}_k := \max\{-u-k, 0\}$ satisfy

$$\|\hat{u}_k\|_{L^{\frac{n}{n-1}}(\Omega)} \le \frac{|\operatorname{spt} \hat{u}_k|}{n\varepsilon\omega_n^{1/n}}, \qquad \|\check{u}_k\|_{L^{\frac{n}{n-1}}(\Omega)} \le \frac{|\operatorname{spt} \check{u}_k|}{n\varepsilon\omega_n^{1/n}}$$

for all $k \ge k_0 := \sup_{\partial\Omega} |\phi|$.

Proof. Using Theorem 3.12 and Lemma 3.13 we see that $\hat{u}_k \in W_0^{1,2}(\Omega)$. Therefore, we can use $\hat{u}_k$ as a test function in the weak formulation of $\operatorname{div} a - \sigma H = 0$. Note that

$$p \cdot a(p) \;=\; \frac{|p|^2}{\sqrt{1+|p|^2}} \;\ge\; |p| \;-\; 1. \tag{5.2}$$

We put $A(k) := \operatorname{spt} \hat{u}_k$ and estimate

$$\begin{aligned}
\|D\hat{u}_k\|_{L^1(\Omega)} \;-\; |A(k)| &= \int_{A(k)} (|D\hat{u}_k| \;-\; 1)\,\mathrm{d}x \overset{(5.2)}{\le} \int_{A(k)} D_i\hat{u}_k \cdot a^i(D\hat{u}_k)\,\mathrm{d}x \\
&= \int_{A(k)} D_i\hat{u}_k \cdot a^i(Du)\,\mathrm{d}x = \int_{A(k)} -\sigma H(\cdot, u, Du)\hat{u}_k \,\mathrm{d}x \le \int_{A(k)} -\sigma H(\cdot, 0, Du)\hat{u}_k\,\mathrm{d}x \\
&\le \left| \int_\Omega H(\cdot, 0, D\hat{u}_k)\hat{u}_k\,\mathrm{d}x \right| \overset{(5.1)}{\le} (1-\varepsilon)\int_\Omega |D\hat{u}_k|\,\mathrm{d}x = (1-\varepsilon)\|D\hat{u}_k\|_{L^1(\Omega)}.
\end{aligned}$$

Finally, we use the Poincaré type inequality from Theorem 3.14 to obtain

$$\|\hat{u}_k\|_{L^{\frac{n}{n-1}}(\Omega)} \le \frac{1}{n\omega_n^{1/n}} \|D\hat{u}_k\|_{L^1(\Omega)} \le \frac{|A(k)|}{n\varepsilon\omega_n^{1/n}}.$$

The second estimate follows by considering $\check{u}_k$ in $A(k) := \operatorname{spt} \check{u}_k$. □

In the next step we present the Lemma of Stampacchia [60].

Lemma 5.8 (**Stampacchia**). *Let $\psi : [k_0, \infty) \to [0, \infty)$ be decreasing, $\gamma > 1$ and suppose that*

$$(h-k)\psi(h) \;\le\; c\big[\psi(k)\big]^{\gamma} \qquad \forall\ h > k \ge k_0.$$

Then for

$$d \;:=\; 2^{\frac{\gamma}{\gamma-1}} c\big[\psi(k_0)\big]^{\gamma-1}$$

we have $\psi(k_0 + d) = 0$.

Proof. Let us put $k_i := k_0 + d - \frac{d}{2^i}$. We apply the inequality with $h = k_{i+1}$ and $k = k_i$ and get

$$\psi(k_{i+1}) \;\le\; \frac{2^{i+1}c}{d}\big[\psi(k_i)\big]^{\gamma}.$$

We will show by induction that

$$\psi(k_i) \;\le\; \frac{\psi(k_0)}{2^{i/(\gamma-1)}}.$$

For $i = 0$ this is clear. Suppose this is true for i. Then

$$\psi(k_{i+1}) \le \frac{2^{i+1}c}{d}\big[\psi(k_i)\big]^{\gamma} \le \frac{2^{i+1}c}{d}\frac{\big[\psi(k_0)\big]^{\gamma}}{2^{i\gamma/(\gamma-1)}} = \frac{\psi(k_0)}{2^{(i+1)/(\gamma-1)}}$$

which finishes the induction and shows that $\lim\limits_{i\to\infty} \psi(k_i) = 0$. Thus, taking the limit of $\psi(k_i) \ge \psi(k_0 + d) \ge 0$ as $i \to \infty$ yields the result. □

Theorem 5.9 (C^0**-estimate**). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^1-boundary and $\phi \in C^0(\partial\Omega)$. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ a solution of* (PMC)$_\sigma$ *for some $\sigma \in [0,1]$ and $H \in C^{0,1}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ increasing in z. If there exists an $\varepsilon \in (0,1]$ such* (5.1) *holds. Then we obtain the following C^0-estimate*

$$\sup_{\Omega} |u| \;\le\; \sup_{\partial\Omega} |\phi| \;+\; \frac{2^{n+1}}{n\varepsilon_0}\left(\frac{|\Omega|}{\omega_n}\right)^{1/n}.$$

Proof. Let $\hat{u}_k := \max\{u - k, 0\}$ for $k \ge k_0 := \sup_{\partial\Omega} |\phi|$ and $A(k) := \operatorname{spt} \hat{u}_k$. Using Proposition 5.7 we obtain for $h > k$

$$(h-k)|A(h)| \le \int_{A(h)} (u-k)\,\mathrm{d}x \le \int_{A(k)} \hat{u}_k\,\mathrm{d}x$$

$$\le\; |A(k)|^{1/n}\|\hat{u}_k\|_{L^{\frac{n}{n-1}}(\Omega)} \le \frac{1}{n\varepsilon\omega_n^{1/n}}|A(k)|^{1+1/n}.$$

Now we set $\psi(k) := |A(k)|$ and apply Stampacchia's Lemma 5.8 with $\gamma = 1 + 1/n$

$$\big|\{x \in \Omega \mid u(x) > d + k_0\}\big| \;=\; 0 \quad \text{for} \quad d := \frac{2^{n+1}}{n\varepsilon}\left(\frac{|A(k_0)|}{\omega_n}\right)^{1/n}.$$

Since u is continuous this implies

$$\sup_{\Omega} u \;\le\; \sup_{\partial\Omega} |\phi| \;+\; \frac{2^{n+1}}{n\varepsilon}\left(\frac{|\Omega|}{\omega_n}\right)^{1/n}.$$

Similarly we can estimate $-u$ using $\check{u}_k := \max\{-u - k, 0\}$ and $\psi(k) := |\operatorname{spt} \check{u}_k|$. This yields the desired result. □

5.2. Interior gradient estimate

As for the C^0-estimate, one can derive theorems which yield an interior gradient estimate (in terms of the gradient at the boundary) for general quasilinear elliptic equations. See for example [23], Theorem 15.2 which implies a result for (PMC) in the case $H = H(x)$ or Theorem 15.6 for equations in divergence form.

The idea is to differentiate the PDE w.r.t. D_k to multiply by $D_k u$ and to sum over k. This yields a PDE for $|Du|^2$. If that PDE satisfies the assumptions of the maximum principle one obtains and estimate for $|Du|$. We will follow this approach for (PMC): Let us use the following definitions

$$a^i(Du) := \frac{D^i u}{\sqrt{1+|Du|^2}}$$

$$a^{ij}(Du) := \frac{\partial a^i}{\partial p^j}\Big|_{Du} = \frac{1}{\sqrt{1+|Du|^2}}\left(\delta^{ij} - \frac{D^i u D^j u}{1+|Du|^2}\right)$$

$$H_{x^k} := \frac{\partial H}{\partial x^k}, \qquad H_z := \frac{\partial H}{\partial z}, \qquad H_{p^k} := \frac{\partial H}{\partial p^k}$$

$$v := \sqrt{1+|Du|^2}.$$

Let us assume that $u \in C^3(\Omega)$. Then we can compute

$$\begin{aligned} 0 &= (D^k u) D_k \left(D_i a^i - H\right) \\ &= (D^k u) D_i \left(a^{ij} D_{jk} u\right) - (D^k u)\left(H_{x^k} + H_z D_k u + H_{p^l} D_{kl} u\right) \\ &= D_i\left(a^{ij} D^k u D_{jk} u\right) - a^{ij} D_i^k u D_{kj} u - \langle Du, H_x\rangle - H_z |Du|^2 - H_{p^l} D^k u D_{kl} u. \end{aligned}$$

Note that $a^{ij} D_{ik} u D_j^k u \geq 0$. Therefore, we obtain for $w := |Du|^2/2$:

$$Lw := D_i(a^{ij} D_j w) - H_{p^l} D_l w \geq -|H_x||Du| + H_z |Du|^2$$

in the case that $H = H(z,p)$ with $H_z \geq 0$ we get $Lw \geq 0$ and the maximum principle Theorem 3.20 is applicable. The same remains true when we replace H by σH. So we obtain the following result.

Proposition 5.10 (**Interior gradient estimate:** $H = H(z,p)$**).** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $u \in C^{2,\alpha}(\overline{\Omega})$ be a solution of* (PMC)$_\sigma$ *with $\sigma \in [0,1]$ and $u \in [-m, M]$. Assume that for some $\beta \in (0,1)$ we have*

$$H \in C^{1,\beta}([-m,M] \times \mathbb{R}^n), \qquad H_z \geq 0.$$

Then the estimate

$$\sup_{\Omega} |Du| \leq \sup_{\partial\Omega} |Du|$$

holds.

Proof. The proof is contained in the computation above. the higher regularity of u follows from the fact that $H \in C^{1,\beta}([-m,M]\times\mathbb{R}^n)$ together with the regularity result for linear equations, Theorem 3.8. □

To treat functions which also depend on x we need to modify the function w to compensate for bad terms of lower order. We make use of the following relations.

Lemma 5.11. *Using the definitions above we have the following relations*

$$a^{ij}D_iuD_ju = v^{-3}|Du|^2, \qquad a^{ij}D_iuD_jv = v^{-4}D^kwD_ku$$

$$a^{ij}D_{ik}uD_j^ku \geq 0, \qquad D_i(a^{ij}D_ju) = v^{-2}\Big(H - 2v^{-3}D^kuD_kw\Big).$$

Proof. Exercise. □

Proposition 5.12 (Interior gradient estimate: $H = H(x,z)$). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $u \in C^{2,\alpha}(\overline{\Omega})$ be a solution of* (PMC)$_\sigma$ *with $\sigma \in [0,1]$ and $u \in [-m,M]$. Assume that for some $\beta \in (0,1)$*

$$H \in C^{1,\beta}(\overline{\Omega}\times[-m,M]), \qquad H_z \geq 0.$$

Then the estimate

$$\sup_{\Omega}|Du| \leq c(1 + \sup_{\partial\Omega}|Du|)$$

holds for some $c = c(m+M, \sup_{\Omega\times[-m,M]}|H|, \sup_{\Omega\times[-m,M]}|H_x|)$. If $H \geq 0$ then the constant does not depend on $|H|$.

Proof. W.l.o.g. we only consider the subdomain where $|Du| > 1$. We set $v := \sqrt{1+|Du|^2}$, $\hat{w} := \log v + f(u)$ and assume that $f' \geq 0$. Using Lemma 5.11 we compute

$$\begin{aligned}
L\hat{w} &:= D_i(a^{ij}D_j\hat{w}) + b^kD_k\hat{w} \\
&:= D_i\left(a^{ij}\left[v^{-2}D_jw + f'D_ju\right]\right) + 2v^{-1}a^{ik}D_iv\left[v^{-2}D_kw + f'D_ku\right] \\
&= v^{-2}D_i(a^{ij}D_jw) + f'D_i(a^{ij}D_ju) + f''a^{ij}D_iuD_ju + 2v^{-1}f'a^{ik}D_ivD_ku \\
&\geq -v^{-2}|H_x||Du| + f'v^{-2}\Big(H - 2v^{-3}D^kuD_kw\Big) + f''v^{-3}|Du|^2 + 2v^{-1}f'a^{ik}D_ivD_ku \\
&\geq v^{-3}\Big(f''|Du|^2 - v|Du||H_x| - v|f'||H|\Big).
\end{aligned}$$

Let us first discuss the case $m = 0$. Using $f(z) := \exp(\mu z)$ we see that

$$L\hat{w} \geq v^{-3}|Du|^2\Big(f'' - 2|H|f' - 2|H_x|\Big) = v^{-3}|Du|^2e^{\mu u}\Big(\mu^2 - 2|H|\mu - 2|H_x|\Big) \geq 0$$

for $\mu := 2(|H|+|H_x|+1)$. Thus, the maximum principle, Theorem 3.20 yields an estimate for $\hat{w}$ which implies

$$\begin{aligned}
\sup_{\Omega}|Du| &\leq \sup_{\Omega}\exp(\hat{w} - f(u)) \\
&\leq \exp\Big(\ln\sup_{\partial\Omega}\sqrt{1+|Du|^2} + 2\exp(\mu M)\Big) \\
&\leq (1 + \sup_{\partial\Omega}|Du|^2)\exp(2\exp(\mu M)).
\end{aligned}$$

To deal with arbitrary values of m we note that $\tilde{u} := u + m \geq 0$ satisfies

$$\operatorname{div}\left(\frac{D\tilde{u}}{\sqrt{1+|D\tilde{u}|^2}}\right) = H(x, \tilde{u} - m) \qquad in\ \Omega.$$

Therefore, the argument above yields an estimate for $|D\tilde{u}| = |Du|$ with M replaced by $M + m$. □

Remark 5.13. The same argument yields an interior gradient estimate for $H = H(x, z, r)$ where r stands for a $|Du|^2$ dependence. In this case we have to assume $H_z \geq 0$ and $H_r \leq 0$. However, note that in this case the boundedness of $|H|$ and $|H_x|$ which appear in the constant is not guaranteed for arbitrary H. So we could not allow for $H = f(x,u)\sqrt{1+|Du|^2}$ even if we impose the conditions $f \leq 0$ and $f_z \geq 0$ (which imply $H_z \geq 0$ and $H_r \leq 0$) because the quantities H and H_x are not bounded.

An extension of the previous proposition to a more geometric dependence on Du is contained in the following theorem.

Theorem 5.14 (**Interior gradient estimate:** $H = H(x, z, \nu)$)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $u \in C^{2,\alpha}(\overline{\Omega})$ be a solution of* (PMC)$_\sigma$ *with $\sigma \in [0,1]$ and $u \in [-m, M]$. Assume that for some $\beta \in (0,1)$*

$$H \in C^{1,\beta}(\overline{\Omega} \times [-m, M] \times \mathbb{R}^{n+1}), \qquad H = H(x, u, \nu(Du)), \qquad H_z \geq 0.$$

Then the estimate

$$\sup_{\Omega} |Du| \leq c\left(1 + \sup_{\partial\Omega} |Du|\right)$$

holds for some $c = c(n, m + M, \sup|H|, \sup|H_x|, \sup|H_\nu|)$ where the supremum is taken over $\Omega \times [-m, M] \times \mathbb{S}^{n+1}$.

Proof. Exercise. □

Exercise II.3 (**Interior gradient estimate:** $H = H(x, z, \nu)$)**.** Prove the interior gradient estimate, Theorem 5.14. Hint: Use the ansatz $\hat{w} := \log v + f(u)$ and compute $L\hat{w} := \Delta_M \hat{w}$ where Δ_M is the Laplace Beltrami operator on $M = \operatorname{graph} u$.

Remark 5.15. The interior estimates we proved shift the problem from the interior to the boundary. So we see that the estimate finally relies on a boundary gradient estimate. One can also prove a purely interior estimate of the kind

$$|Du(0)| \leq c \exp\left(1 + \frac{M^2}{r^2}\right), \qquad u \in C^3(B_r(0)).$$

That is done for example for (PMC) in [64] on just two pages. The advantage is that such an estimate can also be used for solutions $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$. This permits to prove the existence for less regular boundary data $\phi \in C^0(\partial\Omega)$.

5.3. Boundary gradient estimate

The last quantity which remains to be estimated is the gradient of the solution u of $(\text{PMC})_\sigma$ at the boundary. For that purpose we use barriers. The idea of barriers is to find functions $w^\pm$ which have the same boundary values ϕ and lie above and below the solution u. Thus, the gradient of u can be estimates with the help of the gradients of $w^\pm$. We start with a general definition of barriers.

Definition 5.16 (**Definition of Barriers**)**.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let the quasilinear operator Q satisfy the conditions of the comparison principle, Theorem 5.1. Let $d_0 > 0$ and $\mathrm{d} := \mathrm{dist}(\cdot, \partial\Omega)$. We define the boundary strip

$$\Gamma_0 \;:=\; \left\{ x \in \overline{\Omega} \;|\; \mathrm{d}(x) < d_0 \right\}.$$

Let $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ be a solution of

$$Qu = 0 \quad \text{in } \Omega, \qquad u = \phi \quad \text{on } \partial\Omega. \tag{5.3}$$

If there are functions $w^\pm \in C^2(\Gamma_0)$ such that

$$\begin{array}{rcccccll} (i) & w^- & = & u & = & w^+ & \text{on} & \partial\Omega, \\ (ii) & w^- & \le & u & \le & w^+ & \text{on} & \partial\Gamma_0 \cap \Omega, \\ (iii) & Qw^+ & < & 0 & < & Qw^- & \text{in} & \Gamma_0 \cap \Omega \end{array}$$

holds. Then $w^\pm$ are called upper and lower barriers for solutions u of (5.3) in Γ_0.

Remark 5.17. At first, one could think that it is easier to require that (ii) holds in Γ_0 and to ignore (iii). However, since the solution u is unknown it is not clear how to construct $w^\pm$ which satisfy (ii) in all of Γ_0 but also (i). Whereas on $\partial\Gamma_0 \cap \Omega$ this can be realized by demanding $w^+ \ge M$ and $w^- \le -m$ where $u \in [-m, M]$.

The next lemma shows that (ii) and (iii) together with the comparison principle imply the validity of (ii) in all of Γ_0.

Lemma 5.18 (**Application of Barriers**)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2-boundary. Let Q be a quasilinear elliptic Operator satisfying properties $(a) - (c)$ of Theorem* 5.1 *and $\phi \in C^2(\overline{\Omega})$. Furthermore, let $d_0 > 0$ be sufficiently small such that $\mathrm{d} \in C^2(\Gamma_0)$ and that $\|u - \phi\|_{C^0(\overline{\Gamma_0})} \le 1$. Let ψ satisfy*

$$\psi \in C^2([0, d_0]), \qquad \psi(0) = 0, \qquad \psi(d_0) \ge 1. \tag{5.4}$$

If the functions $w^\pm := \pm\psi \circ \mathrm{d} + \phi$ satisfy $\pm Qw^\pm \le 0$ in $\Gamma_0 \cap \Omega$. Then they are barriers for a solution $u \in C^2(\overline{\Omega})$ of (5.3) *in Γ_0 and the estimate*

$$\sup_{\partial\Omega} |Du| \le |\psi'(0)| + \sup_{\partial\Omega} |D\phi|$$

holds.

Proof. Due to the choice of $d_0 > 0$ we see that $w^\pm \in C^2(\Gamma_0)$. Next we observe that $\psi(0) = 0$ implies (i) and $\psi(d_0) \geq 1 \geq \|u - \phi\|_{C^0(\overline{\Gamma})}$ implies (ii). Therefore, we have

$$w^- \leq u \leq w^+ \qquad \text{on } \partial\Gamma_0. \tag{5.5}$$

Now we apply the comparison principle to Q in $\Gamma_0 \cap \Omega$. By condition (iii) of Definition 5.16 and the assumptions on Q, Theorem 5.1 implies that (5.5) holds in $\overline{\Gamma}_0$. Thus,

$$|Du| \leq |Dw^\pm| \leq |\psi'(0) D\,\mathrm{d}\,| + |D\phi|.$$

Since $|D\,\mathrm{d}\,| = 1$ the result follows. □

It turns out that the mean curvature of $\partial\Omega$ will play a role. We use the following convention.

Definition 5.19 (Mean curvature of $\partial\Omega$). Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2-boundary. At $y \in \partial\Omega$ we can choose a coordinate system such that $e_1, \ldots, e_{n-1}$ are tangent to $\partial\Omega$ and e_n is the inward pointing unit normal of $\partial\Omega$. Locally, the boundary can be described as the graph of a C^2-function f in these coordinates. We define

$$H_{\partial\Omega}(y) := \Delta f\big|_y$$

which is the mean curvature of graph f with respect to the lower unit normal to the graph, i.e. the outward pointing unit normal of $\partial\Omega$.

We will make use of a relation between the mean curvature of the boundary and the distance function.

Lemma 5.20. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2-boundary. Then there exists some $d_0 > 0$ such that $\mathrm{d} := \mathrm{dist}(\cdot, \partial\Omega) \in C^2(\Gamma_0)$. Furthermore, for $x \in \Gamma_0$ and $y \in \partial\Omega$ such that $\mathrm{d}(x) = |x - y|$ we have*

$$[D^2\,\mathrm{d}(x)] = diag\left[\frac{-\kappa_1(y)}{1 - \kappa_1(y)\,\mathrm{d}(x)},\ \cdots,\ \frac{-\kappa_{n-1}(y)}{1 - \kappa_{n-1}(y)\,\mathrm{d}(x)},\ 0\right],$$

provided we choose a coordinate system centered at y with basis vectors pointing in the principal directions. Here $\kappa_i(y)$ are the principal curvatures of $\partial\Omega$ in y and $\kappa_i(y) \leq d_0^{-1}$. In particular,

$$\Delta\,\mathrm{d}(x) = -\sum_{i=1}^{n-1} \frac{\kappa_i(y)}{1 - \kappa_i(y)\,\mathrm{d}(x)} \leq -\sum_{i=1}^{n-1} \kappa_i(y) = -H_{\partial\Omega}(y).$$

Proof. Exercise. A proof can be found in [23], Lemma 14.16 and Lemma 14.17. □

Exercise II.4. Try to prove Lemma 5.20. In particular, verify that

$$\Delta\,\mathrm{dist}(x, \partial\Omega) \leq -H_{\partial\Omega}(y)$$

where $y \in \partial\Omega$ is such that $\mathrm{dist}(x, \partial\Omega) = |x - y|$.

Let us first consider the case $H = 0$, i.e. minimal surfaces.

Proposition 5.21 (**Boundary gradient estimate:** $H \equiv 0$)**.** *Let Ω be a bounded domain with C^2-boundary. Let $\phi \in C^2(\overline{\Omega})$. If $u \in C^2(\overline{\Omega})$ satisfies* (PMC)$_\sigma$ *with $H \equiv 0$, i.e. the minimal surface equation and the mean curvature of $\partial\Omega$ is non-negative, i.e. $H_{\partial\Omega} \geq 0$. Then the following a priori estimate holds*

$$\sup_{\partial\Omega} |Du| \leq c\left(d_0, \| \mathrm{d} \|_{C^2(\overline{\Gamma_0})}, \|\phi\|_{C^2(\overline{\Gamma_0})}\right).$$

Proof. Let us choose $d_0 > 0$ sufficiently small (as in Lemma 5.18) and make the ansatz $w^\pm = \pm\psi \circ \mathrm{d} + \phi$. We consider the equivalent operator

$$\overline{Q}u := \overline{a}^{ij}(Du)D_{ij}u := \Big((1+|Du|^2)\delta^{ij} - D^iuD^ju\Big)D_{ij}u = 0.$$

We need to show that $\pm\overline{Q}w^\pm \leq 0$ for a function ψ satisfying (5.4). We start by computing

$$D_i w^\pm = \pm\psi' D_i \mathrm{d} + D_i\phi \tag{5.6}$$

$$|Dw^\pm|^2 = (\psi')^2 \pm 2\psi'\langle D\mathrm{d}, D\phi\rangle + |D\phi|^2 \leq (\psi' + |D\phi|)^2 \tag{5.7}$$

$$D_{ij}w^\pm = \pm\psi'' D_i \mathrm{d}\, D_j \mathrm{d} \pm \psi' D_{ij}\mathrm{d} + D_{ij}\phi. \tag{5.8}$$

We obtain three terms which will be estimated separately:

$$\pm\,\overline{Q}w^\pm = \psi'\overline{a}^{ij}(Dw^\pm)D_{ij}\mathrm{d} + \psi''\overline{a}^{ij}(Dw^\pm)D_i\mathrm{d}\,D_j\mathrm{d} \pm \overline{a}^{ij}(Dw^\pm)D_{ij}\phi.$$

Since $|D\mathrm{d}| = 1$ we see that $D_i\mathrm{d}\,D_{ij}\mathrm{d} = 0$. Together with Lemma 5.20 we obtain

$$\begin{aligned}
&\psi'\overline{a}^{ij}(Dw^\pm)D_{ij}\mathrm{d} \\
&\qquad = \psi'(1+|Dw^\pm|^2)\Delta\mathrm{d} - \psi' D_i\phi D_j\phi D_{ij}\mathrm{d} \\
&\qquad \leq -\psi'(1+|Dw^\pm|^2)H_{\partial\Omega}(y) + c_1\left(\|\phi\|_{C^1(\overline{\Gamma_0})}, \|\mathrm{d}\|_{C^2(\overline{\Gamma_0})}\right)(\psi')^2
\end{aligned}$$

where we assumed that $\psi' \geq 1$. To estimate the second term we assume $\psi'' \leq 0$. Since $\overline{a}^{ij}$ is positive definite we could ignore this term but it will be needed to compensate bad terms. So we compute

$$\begin{aligned}
&\psi''(\mathrm{d})a^{ij}(Dw^\pm)D_i\mathrm{d}\,D_j\mathrm{d} \\
&\qquad = \psi''\Big((1+|Dw^\pm|^2)|D\mathrm{d}|^2 - \langle Dw^\pm, D\mathrm{d}\rangle^2\Big) \\
&\qquad = \psi''\Big(1 + (\psi')^2 \pm 2\psi'\langle D\mathrm{d}, D\phi\rangle + |D\phi|^2 - (\psi' \pm \langle D\mathrm{d}, D\phi\rangle)^2\Big) \\
&\qquad = \psi''\Big(1 + |D\phi|^2 - \langle D\mathrm{d}, D\phi\rangle^2\Big) \\
&\qquad \leq\ \psi''.
\end{aligned}$$

The third term can be estimated by

$$\pm\overline{a}^{ij}(Dw^\pm)D_{ij}\phi \leq \tilde{c}_2\left(\|\phi\|_{C^2(\overline{\Gamma_0})}\right)(1+|Dw^\pm|^2) \leq c_2\left(\|\phi\|_{C^2(\overline{\Gamma_0})}\right)(\psi')^2$$

where we assumed once more that $\psi' \geq 1$. In total we obtain

$$\pm\overline{Q}w^{\pm} \leq -(1+|Dw^{\pm}|^2)\psi' H_{\partial\Omega} + (c_1+c_2)(\psi')^2 + \psi''. \tag{5.9}$$

Let $a, b > 0$. We choose $\psi(d) := a\log(1+bd)$. Note that $\psi(0) = 0$ and $\psi'' = -a^{-1}(\psi')^2 \leq 0$. Since $H_{\partial\Omega} \geq 0$ we obtain

$$\pm\overline{Q}w^{\pm} \leq (c_1+c_2)(\psi')^2 - a^{-1}(\psi')^2 < 0$$

provided $a^{-1} \geq c_1 + c_2$. Furthermore, we need to verify $\psi(d_0) \geq 1$ and $\psi'(d) \geq 1$. All together we need to check that:

$$a\log(1+bd_1) \geq 1, \qquad \frac{ab}{1+bd_1} \geq 1, \qquad a^{-1} \geq c_1 + c_2.$$

where $d_1 \in (0, d_0)$. All this can be achieved for

$$a := \frac{1}{c_1+c_2+1}, \qquad d_1 := \min\{d_0, a/2\}, \qquad b := \max\left\{\frac{1}{a-d_1}, \frac{\exp(a^{-1})-1}{d_1}\right\}$$

and from Lemma 5.18 we obtain the estimate

$$\sup_{\partial\Omega}|Du| \leq ab + \sup_{\partial\Omega}|D\phi| \leq c\left(d_0, \|\,\mathrm{d}\,\|_{C^2(\overline{\Gamma_0})}, \|\phi\|_{C^2(\overline{\Gamma_0})}\right).$$

□

Remark 5.22. Note that the condition $H_{\partial\Omega} \geq 0$ was needed here because the $|Dw^{\pm}|^2$ term is of order $(\psi')^2$ so in total the first term is of order $(\psi')^3$ and we can not absorb it into the good term ψ'' which is only of order $(\psi')^2$. Later we will see that the condition on the mean curvature of $\partial\Omega$ is sharp.

In the next step we extend our barrier construction to the situation where $H = H(x, u)$.

Proposition 5.23 (**Boundary gradient estimate:** $H = H(x, z)$)**.** *Let Ω be a bounded domain with C^2-boundary and $\phi \in C^2(\overline{\Omega})$. Let $u \in C^2(\overline{\Omega})$ be a solution of* (PMC)$_\sigma$. *If $H \in C^1(\overline{\Gamma_0} \times [-R, R])$ for $R := 1 + \sup_{\Gamma_0}|\phi|$ and satisfies $H_z \geq 0$ and the Serrin condition*

$$H_{\partial\Omega}(y) \geq |H(y, \phi(y))| \qquad \forall\, y \in \partial\Omega. \tag{5.10}$$

Then the following a priori estimate holds

$$\sup_{\partial\Omega}|Du| \leq c\left(d_0, \|\,\mathrm{d}\,\|_{C^2(\overline{\Gamma_0})}, \|\phi\|_{C^2(\overline{\Gamma_0})}, \|H\|_{C^1(\overline{\Gamma_0}\times[-R,R])}\right).$$

Proof. Let us choose $d_0 > 0$ sufficiently small (as in Lemma 5.18). We make the ansatz $w^{\pm} = \pm\psi \circ \mathrm{d} + \phi$ and consider the equivalent operator

$$\begin{aligned}\overline{Q_\sigma}u &:= \overline{a}^{ij}(Du)D_{ij}u - \overline{b}(\cdot, u)\\ &:= \left((1+|Du|^2)\delta^{ij} - D^i u D^j u\right)D_{ij}u - \sigma(1+|Du|^2)^{3/2}H(\cdot, u).\end{aligned}$$

We need to show that $\pm\overline{Q}w^{\pm} \leq 0$ for a function ψ satisfying (5.4). Recall the formulae for $w^{\pm}$, i.e. (5.6) and our result for $\overline{a}^{ij}(Du)D_{ij}u.$ in (5.9). We obtain

$$\begin{aligned}\pm\,\overline{Q}w^{\pm} &= \pm\overline{a}^{ij}(Dw^{\pm})D_{ij}w^{\pm} \mp \sigma(1+|Dw^{\pm}|^2)^{3/2}H(\cdot,w^{\pm})\\ &\leq -(1+|Dw^{\pm}|^2)\psi' H_{\partial\Omega}(y) + c(\psi')^2 + \psi'' \mp \sigma(1+|Dw^{\pm}|^2)^{3/2}H(\cdot,w^{\pm})\\ &= -(1+|Dw^{\pm}|^2)\psi'\Big(H_{\partial\Omega}(y) \pm \sigma H(\cdot,w^{\pm})\Big) + c(\psi')^2 + \psi''\\ &\quad \mp \sigma(1+|Dw^{\pm}|^2)\Big(\sqrt{1+|Dw^{\pm}|^2} - \psi'\Big)H(\cdot,w^{\pm}).\end{aligned}$$

Before we continue we note that

$$\begin{aligned}&H_{\partial\Omega}(y) \pm \sigma H(x,w^{\pm}(x))\\ &\quad = H_{\partial\Omega}(y) \pm \sigma H(y,\phi(y)) \mp \sigma\Big[H(y,\phi(y)) - H(x,\phi(y))\Big]\\ &\qquad \mp \sigma\Big[H(x,\phi(y)) - H(x,\phi(x))\Big] \mp \sigma\Big[H(x,\phi(x)) - H(x,\pm\psi\circ \mathrm{d}(x) + \phi(x))\Big].\end{aligned}$$

The last $\mp[\ldots]$-term is always positive since $\psi\circ\mathrm{d} \geq 0$ and $H_z \geq 0$. Together with its pre factor $-(1+|Dw^{\pm}|^2)\psi'$ it is negative so we can drop it. Using $\psi'(d) \geq 1$ we have

$$1+|Dw^{\pm}|^2 \leq c\,(|D\phi|)\,(\psi')^2, \qquad \Big|\sqrt{1+|Dw^{\pm}|^2} - \psi'\Big| \leq 1+|D\phi|.$$

Using the Serrin condition and assuming $|\psi\circ d| \leq 1$ we obtain

$$\begin{aligned}&\pm\,\overline{Q}w^{\pm}\\ &\quad\leq -(1+|Dw^{\pm}|^2)\psi'\Big(H_{\partial\Omega}(y) \pm \sigma H(y,\phi(y))\Big) + c(\psi')^2 + \psi''\\ &\qquad + (1+|Dw^{\pm}|^2)\psi'\Big(\Big|H(y,\phi(y)) - H(x,\phi(y))\Big| + \Big|H(x,\phi(y)) - H(x,\phi(x))\Big|\Big)\\ &\qquad + (1+|Dw^{\pm}|^2)\Big|\sqrt{1+|Dw^{\pm}|^2} - \psi'\Big||H(\cdot,w^{\pm})|\\ &\quad\leq c(\psi')^2 + \psi'' + (1+|Dw^{\pm}|^2)\psi'\Big(\sup_{\Gamma_0\times[-R,R]}|H_x| + \sup_{\Gamma_0\times[-R,R]}|H_z|\sup_{\Gamma_0}|D\phi|\Big)\,\mathrm{d}\\ &\qquad + (1+|Dw^{\pm}|^2)(1+|D\phi|)\sup_{\Gamma_0\times[-R,R]}|H|\\ &\quad\leq c(1+\psi'\,\mathrm{d})(\psi')^2 + \psi''.\end{aligned}$$

We choose $\psi(d) := a\log(1+bd)$. Note that $\psi(0)=0$ and $\psi'' = -a^{-1}(\psi')^2 \leq 0$. Thus

$$\pm\overline{Q}w^{\pm} \leq c(1+\psi'\,\mathrm{d})(\psi')^2 - a^{-1}(\psi')^2 \leq 0$$

provided $\psi'(\mathrm{d})\,\mathrm{d} \leq 1$ and $a^{-1} \geq 2c$. Furthermore, we need to verify $\psi(d_0) \geq 1$ and $\psi'(d) \geq 1$. All together we need to check that:

$$a\log(1+bd) \geq 1, \qquad \frac{abd}{1+bd} \leq 1, \qquad \frac{ab}{1+bd} \geq 1, \qquad a^{-1} \geq 2c.$$

where $d_1 \in (0, d_0)$. All this can be achieved for

$$a := \frac{1}{2c+1}, \qquad d_1 := \min\{d_0, a/2\}, \qquad b := \max\left\{\frac{1}{a-d_1}, \frac{\exp(a^{-1})-1}{d_1}\right\}$$

and from Lemma 5.18 we obtain the estimate

$$\sup_{\partial\Omega} |Du| \leq ab + \sup_{\partial\Omega} |D\phi| \leq c\left(d_0, \|\,\mathrm{d}\,\|_{C^2(\overline{\Gamma_0})}, \|\phi\|_{C^2(\overline{\Gamma_0})}, \|H\|_{C^1(\overline{\Gamma_0}\times[-R,R])}\right)$$

where $R := 1 + \sup_{\Gamma_0} |\phi|$. Note that we used $|w^{\pm}| \leq |\psi \circ \mathrm{d}\,| + |\phi| \leq R$ in Γ_0. □

As for the interior gradient estimate one can extend the result to right hand sides which depend on the unit normal of the graph, i.e.

$$H = H(x, u, \nu(Du)), \qquad \nu(Du) := \frac{1}{\sqrt{1+|Du|^2}} \begin{pmatrix} -Du \\ 1 \end{pmatrix}.$$

Theorem 5.24 (**Boundary gradient estimate:** $H = H(x, z, \nu)$)**.** *Let Ω be a bounded domain with C^2-boundary and $\phi \in C^2(\overline{\Omega})$. Let $u \in C^2(\overline{\overline{\Omega}})$ be a solution of* (PMC)$_\sigma$*. Let $H \in C^1(\overline{\Gamma_0} \times [-R, R] \times \mathbb{R}^{n+1})$ for $R := 1 + \sup_{\Gamma_0} |\phi|$, $H = H(x, z, \nu)$. If satisfies $H_z \geq 0$ and the generalized Serrin condition*

$$H_{\partial\Omega}(y) \geq \left| H\left(y, \phi(y), \begin{pmatrix} \pm\mu(y) \\ 0 \end{pmatrix}\right)\right| \qquad \forall\, y \in \partial\Omega. \tag{5.11}$$

Then the following a priori estimate holds

$$\sup_{\partial\Omega} |Du| \leq c\left(d_0, \|\,\mathrm{d}\,\|_{C^2(\overline{\Gamma_0})}, \|\phi\|_{C^2(\overline{\Gamma_0})}, \|H\|_{C^1(\overline{\Gamma_0}\times[-R,R]\times[-1,1]^{n+1})}\right).$$

Proof. Exercise. □

Exercise II.5. Try to prove Theorem 5.24, i.e. a boundary gradient estimate in the case where $H = H(x, z, \nu)$. Hint: Try a proof along the lines of the proof of Theorem 5.23. Show first that

$$\left| H\Big(x, \phi(x), \nu(Dw^{\pm})\Big) - H\left(y, \phi(y), \begin{pmatrix} \pm\mu(y) \\ 0 \end{pmatrix}\right)\right| \leq c_1\,\mathrm{d}(x) + \frac{c_2}{\psi'(\mathrm{d}(x))}.$$

where $y \in \partial\Omega$ is such that $\mathrm{d}(x) = |x - y|$.

5.4. Existence and uniqueness theorem

Before we state the existence theorem let us remark that no additional assumption is needed for the C^0-estimate for surfaces of constant mean curvature (CMC).

Proposition 5.25 (C^0**-estimate for CMC surfaces**)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{2,\alpha}$ boundary. Let $u \in C^2(\overline{\Omega})$ be a solution of* (PMC) *with constant right hand side $H \in \mathbb{R}$ and boundary value $\phi \in C^2(\overline{\Omega})$. If $H_{\partial\Omega} \geq |H|$ then the estimate*

$$\sup_{\overline{\Omega}} |u| \leq \sup_{\partial\Omega} |\phi| + e^{\operatorname{diam}(\Omega)(1+|H|)}$$

holds.

Proof. We make the ansatz

$$v : \overline{\Omega} \to \mathbb{R} : x \mapsto \sup_{\partial\Omega} |\phi| + \frac{e^{\mu\delta}}{\mu}\left(1 - e^{-\mu\,\mathrm{d}(x)}\right), \qquad \mu := 1 + |H|, \quad \delta := \mathrm{diam}(\Omega).$$

Using $H_{\partial\Omega} \geq |H|$ together with Lemma 5.20 we obtain

$$\Delta\,\mathrm{d} \leq -H_{\partial\Omega} \leq -|H|.$$

This estimate is valid on the subset $\Omega_1 \subset \Omega$ consisting of points having a unique closest point on $\partial\Omega$. On Ω_1 we compute

$$\begin{aligned}\overline{a}^{ij} D_{ij} v &:= \left((1 + |Dv|^2)\delta^{ij} - D^i v D^j v\right) D_{ij} v = |Dv|\left[(1 + |Dv|^2)\Delta\,\mathrm{d} - \mu\right] \\ &\leq -|Dv|\left[(1 + |Dv|^2)|H| + 1 + |H|\right] \leq -|Dv|(2 + |Dv|^2)|H| \\ &\leq -(1 + |Dv|^2)^{3/2}|H| \leq (1 + |Dv|^2)^{3/2} H.\end{aligned}$$

Thus

$$\overline{Q}v := \overline{a}^{ij} D_{ij} v - (1 + |Dv|^2)^{3/2} H \leq 0 = \overline{Q}u \quad \text{ in } \Omega_1.$$

From the proof the the quasilinear comparison principle we see that this allows us to construct a linear operator L such that $L(u - v) \geq 0$ in Ω_1. Therefore, the maximum principle for linear operators implies that the maximum of $u - v$ is attained on $\partial\Omega$ or in $\Omega \setminus \Omega_1$.

Let us assume that the maximum is attained at a point $x \in \Omega \setminus \Omega_1$. Let $y \in \partial\Omega$ be a closest point to x on $\partial\Omega$. We put $\gamma : [0, 1] \to \mathbb{R} : t \mapsto \gamma(t) := ty + (1 - t)x$. If $w = u - v$ has a maximum at x then

$$\frac{d^+}{dt}\Big|_{t=0} (w \circ \gamma)(t) = \langle Du(x), y - x\rangle - e^{\mu(\delta - \mathrm{d})} \frac{d^+}{dt}\Big|_{t=0} (\mathrm{d} \circ \gamma)(t) \leq 0.$$

Therefore, we must have $Du(x) \neq 0$. On the other hand, starting from x the function v is monotone decreasing along γ. This shows that if w has a maximum at x also $u = v + w$ must have a maximum at x and therefore $Du(x) = 0$. This contradiction shows that the maximum must be attained on $\partial\Omega$. □

Finally, we can put together all a priori estimates in order to obtain an existence result for (PMC).

Theorem 5.26 (**Existence and Uniqueness for** (PMC))**.** *Let $n \geq 2$, $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{2,\alpha}$-boundary and $\phi \in C^{2,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$. Let $H \in C^{1,\beta}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^{n+1})$, $H = H(x, u, \nu(Du))$.Suppose that H satisfies $H_z \geq 0$ and*

$$H_{\partial\Omega}(y) \geq \left| H\left(y, \phi(y), \begin{pmatrix} \pm\mu(y) \\ 0 \end{pmatrix}\right)\right| \qquad \forall\, y \in \partial\Omega. \tag{5.12}$$

If H is not constant assume also that for some $\varepsilon > 0$

$$\left| \int_\Omega H(\cdot, 0, \nu(D\eta))\eta\, dx \right| \leq (1 - \varepsilon) \int_\Omega |D\eta|\, dx \qquad \forall\, \eta \in C_0^1(\Omega). \tag{5.13}$$

Then the Dirichlet problem of prescribed mean curvature (PMC) *has a unique solution $u \in C^{2,\alpha}(\overline{\Omega})$.*

Proof. All conditions of Corollary 4.10 are satisfied. Therefore, the existence proof is reduced to a priori estimates in $C^1(\overline{\Omega})$ for solutions $u \in C^{2,\alpha}(\overline{\Omega})$ of $(PMC)_\sigma$. If $H = const.$ the boundedness of $|u|$ follows from Proposition 5.25. Otherwise, this bound follows from (5.13) together with Theorem 5.9. The gradient can be estimated in the interior with the help of Theorem 5.14. Finally, the boundary gradient estimate is guaranteed by (5.12) and Theorem 5.24. Uniqueness follows from Corollary 5.3 together with $H_z \geq 0$. □

Remark 5.27 (**Sharpness of the Serrin condition**)**.** Inequality (5.12) is called Serrin condition. It is sharp in the following sense: If $H \in C^1(\overline{\Omega})$ is either non-positive or non-negative and

$$H_{\partial\Omega}(y) < |H(y)| \qquad \text{for some } y \in \partial\Omega.$$

Then for any $\delta > 0$ there exists some $\phi \in C^\infty(\overline{\Omega})$ with $\sup_{\overline{\Omega}} |\phi| < \delta$ such that (PMC) has no solution. See [23], Corollary 14.13 and (14.86). For the minimal surface equation, i.e. $H \equiv 0$ this yields solvability for all $\phi \in C^{2,\alpha}(\overline{\Omega})$ if and only if $H_{\partial\Omega} \geq 0$, i.e. in mean convex domains.

Remark 5.28. The dependence of H on the gradient via the normal was first investigated by Bergner [4] who obtained a result in convex domains. The boundary gradient estimate via the generalized Serrin condition (which also allows non-convex domains) was obtained by the author [42] in his diploma thesis under the supervision of Gerhard Huisken.

6. General theory for fully nonlinear problems

The main tool in the existence proof will be a Banach space version of the inverse function theorem. In order to state it we recall the following definition.

Definition 6.1 (Differentiability in normed spaces). Let X, Y and Z be normed spaces and $\tilde{X} \subset X$ open. A map $T : \tilde{X} \to Z$ is called Gâteaux-differentiable at $x_0 \in \tilde{X}$ if there exists a bounded linear map $L_{x_0} \in \mathcal{L}(X, Z)$ such that

$$\lim_{t \to 0} \frac{T(x_0 + tx) - T(x_0)}{t} = L_{x_0} x \qquad \forall\, x \in X.$$

T is called Fréchet-differentiable at $x_0 \in \tilde{X}$ if the convergence is uniform in $x \in B_X$, i.e.

$$\frac{\|T(x_0 + x) - T(x_0) - L_{x_0}(x)\|_Z}{\|x\|_X} \to 0, \qquad as\ \|x\|_X \to 0.$$

T is called Gâteaux- (or Fréchet)-differentiable in $\tilde{X}$ if the corresponding statement holds for all $x_0 \in \tilde{X}$. We use the classical notation

$$DT : \tilde{X} \to \mathcal{L}(X, Z) : x \mapsto DT\big|_x := L_x$$

The map $x \mapsto L_x$ is denoted by DT.

Suppose that $T : X \times Y \to Z$ is Gâteaux- (or Fréchet)-differentiable at (x_0, y_0). Then the partial Gâteaux- (or Fréchet)-derivatives are bounded linear maps $L^1_{(x_0,y_0)} : X \to Z$ and $L^2_{(x_0,y_0)} : Y \to Z$ such that

$$L_{(x_0,y_0)}(x, y) = L^1_{(x_0,y_0)}(x) + L^2_{(x_0,y_0)}(y).$$

Here we use the notation D_1T or D_xT and similarly D_2T or D_yT.

Remark 6.2. The Fréchet derivative is the generalization of the total derivative of a map $T : U \subset \mathbb{R}^n \to \mathbb{R}^m$. The Gâteaux derivative is the analogue of the directional derivative. Note that we have the classical properties (linearity, chain rule, etc.). In particular, if T is Gâteaux-differentiable and DT) is continuous, then T is Fréchet-differentiable, more precisely continuously Fréchet-differentiable.

Let us now state a version of the inverse function theorem in Banach spaces.

Theorem 6.3 (Inverse function theorem for Banach spaces). *Let X, Z be Banach spaces and $\tilde{X} \subset X$ be open. Suppose that $T : \tilde{X} \times \mathbb{R} \to Z$ is continuously Fréchet-differentiable at (x_0, σ_0) with $T(x_0, \sigma_0) = 0$ and that the partial Fréchet-derivative $D_1T\big|_{(x_0,\sigma_0)}$ is invertible[1]. Then there exists $\varepsilon > 0$ such that for all $\sigma \in (\sigma_0 - \varepsilon, \sigma_0 + \varepsilon)$ there exists $x_\sigma \in \tilde{X}$ with $T(x_\sigma, \sigma) = 0$.*

[1] Since X and Y are Banach spaces and $L := D_1T\big|_{(x_0,\sigma_0)} \in \mathcal{L}(X, Z)$ the bijectivity of L implies already that L is an isomorphism, i.e. that L^{-1} is continuous. This follows from the open mapping theorem.

Proof. See, [23], Theorem 17.6. □

Remark 6.4 (Nash-Moser inverse function theorem). In some problems it might happen that one loses regularity, i.e. that the solutions of a second order problem have not necessarily two degrees of regularity more than the right hand side. In this situation one can not use the inverse function theorem in the Banach space setting. However, there is a version by Nash and Moser (explained in detail by Hamilton [26]) which allows us to replace the Banach spaces by C^∞. Nash used this tool first in [49] for his theorem on isometric embeddings. Later the result was applied by Hamilton in [27] to give a first prove of the well-posedness of Ricci flow[2].

6.1. Fully nonlinear Dirichlet problems

Definition 6.5 (Nonlinear Dirichlet problem). Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We denote by $Sym(\mathbb{R}^n)$ the space of symmetric real valued matrices and consider the continuous map

$$F : \Gamma := \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times Sym(\mathbb{R}^n) \to \mathbb{R} : (x, z, p, q) \mapsto F(x, z, p, q).$$

Assume that F is differentiable. Then we put

$$F_{q^{ij}} := \frac{\partial F}{\partial q^{ij}}, \qquad F_{p^k} := \frac{\partial F}{\partial p^k}, \qquad F_z := \frac{\partial F}{\partial z}$$

and use the short cut $[u] := (\cdot, u, Du, D^2u)$. We say that F is elliptic in $A \subset \Gamma$ if the matrix $[F_{q^{ij}}]$ is positive definite in A. Furthermore, F is called uniformly elliptic in A if

$$0 < \lambda_{\min} \leq F_{q^{ij}}\xi_i\xi_j \leq \Lambda_{\max} < \infty \qquad \forall\ \xi \in \mathbb{S}^n$$

Finally, F is called (uniformly) elliptic with respect to $u \in C^2(\overline{\Omega})$ if F is (uniformly) elliptic in $[u(x)]$ for all $x \in \overline{\Omega}$. We formulate the corresponding Dirichlet problem

$$\begin{cases} F[u] := F(\cdot, u, Du, D^2u) = 0 & \text{in } \Omega, \\ u = \phi & \text{on } \partial\Omega. \end{cases} \tag{6.1}$$

for $\phi \in C^{2,\alpha}(\overline{\Omega})$.

As for quasilinear equations we can formulate a comparison principle.

Theorem 6.6 (Comparison principle for fully nonlinear operators). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let F be as in Definition 6.5. Furthermore, we assume that*

(a) F is continuously differentiable w.r.t. z, p and q in Γ,

(b) F is elliptic with respect to $tu + (1-t)v$ for all $t \in [0, 1]$,

(c) F is non-increasing in z.

If $u, v \in C^2(\overline{\Omega})$ satisfy $F[u] \geq F[v]$ in Ω and $u \leq v$ on $\partial\Omega$. Then $u \leq v$ in $\overline{\Omega}$.

[2] Later the short-time existence result for Ricci flow was simplified by DeTurk [9] and is now called the DeTurk trick.

Exercise II.6. Try to prove the Comparsion principle for fully nonlinear operators, Theorem 6.6 along the lines of the proof for the corresponding Theorem for quasilinear operators, Theorem 5.1.

We will apply the method of continuity together with the inverse function theorem to prove existence for the fully nonlinear problems.

Theorem 6.7 (**Existence criterion for fully nonlinear Dirichlet problems**)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{2,\alpha}$-boundary. Let $U \subset C^{2,\alpha}(\overline{\Omega})$ be open and $\phi \in U$. Suppose that*

$$F, F_z, F_{p^k}, F_{q^{ij}} \in C^{0,\alpha}(\overline{\Gamma}), \qquad F_z, F_{p^k}, F_{q^{ij}} \in C^{0,1} \text{ w.r.t. } q.$$

Furthermore, assume that with respect to all function u in

$$\mathcal{M} := \{\ u \in U \mid \exists \sigma \in [0,1] : F[u] = (1-\sigma)F[\phi] \text{ in } \Omega, \quad u = \phi \text{ on } \partial\Omega\ \}$$

$F_z[u] \leq 0$ and F is uniformly elliptic. If $\overline{\mathcal{M}} \subset U$ and $\mathcal{M}$ is bounded in $C^{2,\alpha}(\overline{\Omega})$. Then the fully nonlinear Dirichlet problem (6.1) *is solvable in U.*

Proof. Let us consider $v := u - \phi$, i.e. we want to solve

$$F[v + \phi] = 0 \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega, \qquad v + \phi \in U \tag{6.2}$$

We want to rephrase the problem as $T(v, 1) = 0$. Therefore, we use

$$X := \{v \in C^{2,\alpha}(\overline{\Omega}) \mid v = 0 \text{ on } \partial\Omega\}, \qquad Z := C^{0,\alpha}(\overline{\Omega})$$

$$\tilde{X} := \{v \in X \mid v + \phi \in U\}$$

and define

$$T : \tilde{X} \times \mathbb{R} \to Z : (v, \sigma) \mapsto T(v, \sigma) := F[v + \phi] - (1 - \sigma)F[\phi].$$

Obviously, finding $v \in \tilde{X}$ such that $T(v, 1) = 0$ is equivalent to solving (6.2). Furthermore, we see that $T(v, 0) = 0$ has the trivial solution $v = 0$. Therefore, the set

$$I := \{\ \sigma \in [0,1] \mid \exists v \in \tilde{X} : T(v, \sigma) = 0\ \}$$

is not empty. If we can show that it is at the same time open and closed we get $I = [0, 1]$. So in particular $1 \in I$ which is the desired result.

I is open: Let $\sigma_0 \in I$ and $v_0 \in \tilde{X}$ such that $T(v_0, \sigma_0) = 0$. We want to apply the inverse function theorem, Theorem 6.3 to T. Due to our regularity assumptions for F we see that T has a continuous Gâteaux derivative. This implies that T is continuously Fréchet-differentiable. The partial Fréchet derivative with respect to the first variable at any (v_0, σ_0) is given by

$$L : X \to Z : v \mapsto Lv := F_{q^{ij}}[v_0]D_{ij}v + F_{p^k}[v_0]D_k v + F_z[v_0]v.$$

Since L has Hölder continuous coefficiens, is uniformly elliptic and satisfies $F_z[u] \leq 0$, Theorem 3.6, yields the invertibility of L. Thus, the inverse function theorem, Theorem

6.3, implies the solvability of $T(v,\sigma)=0$ for all σ in a small neighborhood of σ_0.

I is closed: Let $(\sigma_k)_{k\in\mathbb{N}} \subset I$ converge to some $\sigma \in \mathbb{R}$. Let $(v_k)_{k\in\mathbb{N}} \subset \tilde{X}$ be such that $T(v_k,\sigma_k)=0$. Then $u_k := v_k+\phi \in \mathcal{M}$. By assumption $\|u_k\|_{C^{2,\alpha}(\overline{\Omega})} \leq C$. Therfore, by Arzelà-Ascoli, the sequence[3] converges in $C^2(\overline{\Omega})$ to some function $u = v+\phi \in C^{2,\alpha}(\overline{\Omega})$ and $T(v,\sigma)=0$. Since $\overline{\mathcal{M}} \subset U$ we know that $u \in U$ and therefore $v \in \tilde{X}$. Thus, $\sigma \in I$. ☐

Remark 6.8 (Why the subset $\tilde{X}$ might be necessary). Usually one would first try to apply the theorem with $U = C^{2,\alpha}(\overline{\Omega})$. In this case $\overline{\mathcal{M}} \subset U$ is trivially satisfied. However, the conditions on F need to be verified on a larger set $\mathcal{M}$. That this is not always possible can be seen from the following example: The equation of prescribed Gauss curvature takes the form

$$F[u] := \det(D^2u) - K(x,u,Du)(1+|Du|^2)^{(n+2)/2} = 0.$$

Here F will only be elliptic w.r.t. $u \in C^{2,\alpha}(\overline{\Omega})$ if u is in the subset $U \subset C^{2,\alpha}(\overline{\Omega})$ of uniformly convex functions.

6.2. Fully nonlinear oblique derivative problems

Definition 6.9 (Nonlinear oblique derivative problem). Let F be as in Definition 6.5 and let

$$G : \Gamma' := \partial\Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} : (x,z,p) \mapsto G(x,z,p), \qquad G_z := \frac{\partial G}{\partial z}, \qquad G_{p^k} := \frac{\partial G}{\partial p^k}.$$

If $G \in C^1(\Gamma')$ and $\partial\Omega \in C^1$ we say that G is oblique in $A \subset \Gamma'$ if

$$G_{p^k}\mu_k > 0 \quad \text{in } A, \qquad \mu: \text{ exterior unit normal of } \partial\Omega.$$

Furthermore, G is called oblique with respect to $u \in C^2(\overline{\Omega})$ if G is oblique in $\{(x,z,p) \in \Gamma' \mid (z,p) = (u(x),Du(x))\}$. We formulate the corresponding boundary value problem problem

$$\begin{cases} F[u] := F(\cdot,u,Du,D^2u) = 0 & \text{in } \Omega, \\ G[u] := G(\cdot,u,Du) = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.3}$$

We obtain the Dirichlet problem for $G = G(x,z) := z - \phi(x)$ but since in that situation G does not need to be oblique we make the distinction between Dirichlet and oblique derivative problems.

Remark 6.10. Note that if F is an affine linear function in the q-variable the Dirichlet problem is quasilinear whereas the oblique derivative problem still might have a fully nonlinear behavior in the boundary operator.

For the fully nonlinear oblique derivative problem we obtain the following result

[3]The theorem of Arzelà-Ascoli just yields a converging subsequence but then one can argue that the whole sequence has to converge.

Theorem 6.11 ($C^{2,\alpha}$**-criterion for fully nonlinear oblique derivative problems).** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{2,\alpha}$-boundary. Let $U \subset C^{2,\alpha}(\overline{\Omega})$ be open and $\psi \in U$. Suppose that*

$$F, F_z, F_{p^k}, F_{q^{ij}} \in C^{0,\alpha}(\overline{\Gamma}), \quad F_z, F_{p^k}, F_{q^{ij}} \in C^{0,1} \text{ w.r.t. } q, \quad G, G_z, G_{p^k} \in C^{1,\alpha}(\overline{\Gamma'}).$$

Furthermore, assume that with respect to all functions u in

$$\mathcal{M} := \Big\{ \; u \in U \mid \exists \sigma \in [0,1] : F[u] = (1-\sigma)F[\psi] \text{ in } \Omega \\ G[u] = (1-\sigma)G[\psi] \text{ on } \partial\Omega \; \Big\}$$

F is uniformly elliptic, G is oblique and

$$F_z\big|_{[u]} \leq 0, \qquad G_z\big|_{[u]} \geq 0$$

with at least one inequality being strict. If $\overline{\mathcal{M}} \subset U$ and $\mathcal{M}$ is bounded in $C^{2,\alpha}(\overline{\Omega})$. Then the fully nonlinear oblique derivative problem (6.3) *is solvable in U.*

Proof. We want to rephrase the problem (6.3) as $T(v,1) = 0$. Therefore, we set

$$X := C^{2,\alpha}(\overline{\Omega}), \qquad \tilde{X} := U, \qquad Z := C^{0,\alpha}(\overline{\Omega}) \times C^{1,\alpha}(\partial\Omega)$$

and denote the restriction to the boundary $\partial\Omega$ of a function $f : \overline{\Omega} \to \mathbb{R}$ by $\check{f}$. Then we define

$$T : \tilde{X} \times \mathbb{R} \to Z : (u,\sigma) \mapsto T(u,\sigma) := \Big(F[u] - (1-\sigma)F[\psi] \; , \; G[\check{u}] - (1-\sigma)G[\check{\psi}] \Big).$$

Obviously, finding $u \in \tilde{X}$ such that $T(u,1) = 0$ is equivalent to solving (6.3). Furthermore, we see that $T(u,0) = 0$ has the solution $u = \psi \in \tilde{X}$. Therefore, the set

$$I := \{ \; \sigma \in [0,1] \mid \exists u \in \tilde{X} : T(u,\sigma) = 0 \; \}$$

is not empty. If we can show that it is at the same time open and closed we get $I = [0,1]$. So in particular $1 \in I$ which is the desired result.

I is open: Let $\sigma_0 \in I$ and $u_0 \in \tilde{X}$ such that $T(u_0,\sigma_0) = 0$. We want to apply the inverse function theorem, Theorem 6.3 to T. Due to our regularity assumptions for F an G we see that T has a continuous Gâteaux derivative. This implies that T is continuously Fréchet-differentiable. The partial Fréchet derivative with respect to the first variable at any (u_0,σ_0) is given by

$$(Lu, Nu) := \Big(F_{q^{ij}}[u_0]D_{ij}u + F_{p^k}[u_0]D_k u + F_z[u_0]u \; , \; G_z[\check{u}_0]u + G_{p^k}[\check{u}_0]D_k u \Big)$$

Since L has $C^{0,\alpha}$-coefficiens, is uniformly elliptic and satisfies $F_z[u] \leq 0$ and N as $C^{1,\alpha}$-coefficients satisfies $G_z[\check{u}_0] \geq 0$ and is oblique, Theorem 3.10, yields the invertibility of (L,N).

I is closed: Let $(\sigma_k)_{k\in\mathbb{N}} \subset I$ converge to some $\sigma \in \mathbb{R}$. Let $(u_k)_{k\in\mathbb{N}} \subset \tilde{X}$ be such that $T(u_k,\sigma_k) = 0$. Then $u_k \in \mathcal{M}$ and by assumption $\|u_k\|_{C^{2,\alpha}(\overline{\Omega})} \leq C$. Using the Arzelà-Ascoli theorem we see that the sequence is converging in C^2 to some function $u \in C^{2,\alpha}(\overline{\Omega})$ and $T(u,\sigma) = 0$. Since $\overline{\mathcal{M}} \subset U$ we know that $u \in \tilde{X}$ and thus $\sigma \in I$. □

Remark 6.12. Note that compared to the existence criterion for *quasilinear* Dirichlet problems, Theorem 4.7 the existence criterion for *fully nonlinear* Dirichlet problems requires a priori estimates in $C^{2,\alpha}(\overline{\Omega})$ instead of $C^{1,\alpha}(\overline{\Omega})$. For the quasilinear Dirichlet problem it is obvious that one can reduce to $C^{1,\alpha}(\overline{\Omega})$-estimates by applying the linear Schauder theory. But even if F is quasilinear this procedure does not work for fully nonlinear G. However, also in the situation

$$\begin{cases} F[u] := a^{ij}(\cdot,u,Du)D_{ij}u + a(\cdot,u,Du) = 0 & \text{in } \Omega \\ G[u] := b(\cdot,u,Du) + b^i(\cdot,u)D_iu + b_0(\cdot,u) + \phi = 0 & \text{on } \partial\Omega \end{cases} \tag{6.4}$$

we can reduce to $C^{1,\alpha}$-estimates.

Theorem 6.13 ($C^{1,\beta}$**-criterion for quasilinear oblique derivative problems**). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{2,\alpha}$-boundary and $\psi \in C^{2,\alpha}(\overline{\Omega})$. Suppose that*

$$a^{ij}, a^{ij}_z, a^{ij}_{p^k}, a, a_z, a_{p^k} \in C^{0,\alpha}(\overline{\Gamma})$$

and

$$b, b_z, b_{p^k} \in C^{1,\alpha}(\overline{\Gamma'}), \qquad b_0, (b_0)_z, b^i_z \in C^{1,\alpha}(\overline{\Omega}\times\mathbb{R}), \qquad \phi \in C^{1,\alpha}(\overline{\Omega}).$$

Furthermore, assume that with respect to all functions u in

$$\mathcal{M} := \Big\{\ u \in C^{2,\alpha}(\overline{\Omega}) \mid \exists \sigma \in [0,1] : F[u] = (1-\sigma)F[\psi] \text{ in } \Omega \\ G[u] = (1-\sigma)G[\psi] \text{ on } \partial\Omega\ \Big\}$$

F is uniformly elliptic, G is oblique and

$$F_z\big|_{(\cdot,u,Du)} \le 0 \quad \text{in } \Omega, \qquad G_z\big|_{(\cdot,u,Du)} \ge 0 \quad \text{on } \partial\Omega$$

where at least one of these inequalities is strict. If for some $\beta \in (0,1)$ the set $\mathcal{M}$ is bounded in $C^{1,\beta}(\overline{\Omega})$. Then the problem (6.4) *is solvable in $C^{2,\alpha}(\overline{\Omega})$.*

Sketch of proof. Due to the proof of Theorem 6.11 it remains to show that a sequence $(u_k)_{k\in\mathbb{N}} \subset C^{2,\alpha}(\overline{\Omega})$ of solutions of $T(u_k,\sigma_k) = 0$ which is uniformly bounded in $C^{1,\beta}(\overline{\Omega})$ is automatically uniformly bounded in $C^{2,\alpha}(\overline{\Omega})$. To achieve that goal we put $w := u_k - u_l$. Starting from $F[u_k] - F[u_l] = 0$ we obtain

$$Lw := a^{ij}[u_k]D_{ij}w = f := \Big(a^{ij}[u_l] - a^{ij}[u_k]\Big)D_{ij}u_l + a[u_l] - a[u_k].$$

Similarly, starting from $G[u_k] - G[u_l] = 0$ we obtain

$$Nw := \left\{\int_0^1 G_{p^k}\big|_{(\cdot,u_k,tDu_k+(1-t)Du_l)}dt\right\} D_kw = \phi := G[u_l] - G(\cdot,u_k,Du_l).$$

Using our assumptions on the regularity of the coefficients together with the a priori estimates of the linear theory and an interpolation inequality for Hölder spaces one can show (see [23], Lemma 17.29 or [37], Chapter 10, Theorem 1.2) that the following estimate holds

$$\|w\|_{C^{2,\alpha}(\overline{\Omega})} \leq c\left(1 + \|u_k\|_{C^{2,\alpha}(\overline{\Omega})}\right)\|w\|_{C^1(\overline{\Omega})} \tag{6.5}$$

with uniformly bounded constant c. Now assume that $(u_k)_{r\in\mathbb{N}}$ is not uniformly bounded in $C^{2,\alpha}(\overline{\Omega})$. Then there exists a subsequence $u_i := u_{k_i}$ such that $2\|u_i\|_{C^{2,\alpha}(\overline{\Omega})} \leq \|u_{i+1}\|_{C^{2,\alpha}(\overline{\Omega})}$. Now we put $w_i := u_i - u_{i-1}$ to obtain

$$\|u_i\|_{C^{2,\alpha}(\overline{\Omega})} \leq 2\|w_i\|_{C^{2,\alpha}(\overline{\Omega})} \leq 2c\left(1 + \|u_i\|_{C^{2,\alpha}(\overline{\Omega})}\right)\|w_i\|_{C^1(\overline{\Omega})}.$$

Dividing by $(1+\|u_i\|_{C^{2,\alpha}(\overline{\Omega})})$ the left hand side never vanishes whereas the right hand side tends to zero. this is a contradiction. Thus, the $C^{2,\alpha}$-norms are uniformly bounded and inequality (6.5) even yields convergence in $C^{2,\alpha}(\overline{\Omega})$. For uniqueness see Exercise II.7 □

Remark 6.14. In order to establish the reduction from $C^{2,\alpha}$-estimates to $C^{1,\beta}$ estimates it suffices to require

$$a^{ij}, a \in C^{0,\alpha}(U_K), \quad b, b_{p^k} \in C^{1,\alpha}(U_K), \quad b_0, b^i \in C^{1,\alpha}(\overline{\Omega}\times[-K,K]), \quad \phi \in C^{1,\alpha}(\overline{\Omega})$$

where $U_K := \overline{\Omega}\times[-K,K]\times[-K,K]^n$ and $K := \|u\|_{C^1(\overline{\Omega})}$.

Exercise II.7. Which assumptions in addition to the assumptions in Theorem 6.13 are needed to prove that there exists at most one solution $u \in C^{2,\alpha}(\overline{\Omega})$ of the quasilinear oblique derivative problem?

Let us specialize further by requiring F to have divergence form and G to be a co-normal Neumann condition, i.e.

$$\begin{cases} F[u] := D_i\left(a^i(\cdot, u, Du)\right) + a(\cdot, u, Du) = 0 & \text{in } \Omega \\ G[u] := a^i(\cdot, u, Du)\mu_i + \phi = 0 & \text{on } \partial\Omega \end{cases} \tag{6.6}$$

where μ is the outward unit normal of $\partial\Omega$. In this case the reduction to C^1-apriori estimates is very similar to our proof for the corresponding Dirichlet problem.

Theorem 6.15 (**C^1-criterion for conormal oblique derivative problems**)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{2,\alpha}$-boundary and $\psi \in C^{2,\alpha}(\overline{\Omega})$. Suppose that*

$$a^i, a^i_z, a^i_{p^k} \in C^{1,\alpha}(\overline{\Gamma}), \quad a, a_z, a_{p^k} \in C^{0,\alpha}(\overline{\Gamma}), \quad \phi \in C^{1,\alpha}(\overline{\Omega}).$$

Furthermore, assume that with respect to all functions u in

$$\mathcal{M} := \left\{ u \in C^{2,\alpha}(\overline{\Omega}) \mid \exists\sigma\in[0,1] : \begin{array}{l} F[u] = (1-\sigma)F[\psi] \text{ in } \Omega \\ G[u] = (1-\sigma)G[\psi] \text{ on } \partial\Omega \end{array} \right\}$$

F is uniformly elliptic, G is oblique and

$$F_z\big|_{(\cdot,u,Du)} \leq 0 \quad \text{in } \Omega, \qquad a^i_z\big|_{(\cdot,u,Du)}\mu_i \geq 0 \quad \text{on } \partial\Omega$$

where at least one of these inequalities is strict. If the set $\mathcal{M}$ is bounded in $C^1(\overline{\Omega})$. Then the problem (6.6) is solvable in $C^{2,\alpha}(\overline{\Omega})$.

Sketch of proof. The proof of the a priori Hölder gradient estimate is similar to the corresponding proof for the Dirichlet problem. The interior estimate remains unchanged. For the boundary estimate again a diffeomorphism ψ is used to locally flatten the boundary and to reduce the proof to finding estimates for $v := u \circ \psi^{-1}$.

As before, the estimate for $w = D_n v$ follows from the Morrey estimates for $w = D_k v$ with $1 \leq k \leq n-1$ by solving the PDE w.r.t. $D_{nn}v$. To prove the Morrey estimate for $w = D_k v$ with $1 \leq k \leq n-1$ we proceed as for the Dirichlet boundary problem: We use the fact that w is the weak solution of a linear equation and integrate against test functions $\eta^{\pm} = \rho^2(\pm w - c)$. Using the ellipticity together with Young's inequality and an integration by parts (to get rid of the boundary term coming from the Neumann condition) we arrive at

$$\int_{A_{k,R}} |Dw|^2 \rho^2 \, dy \leq c \int_{A_{k,R}} (w-c)^2 |D\rho|^2 + c|A_{k,R}| \tag{6.7}$$

where $A_{k,R} := B_R(y_0) \cap D^+ \cap \{w > k\}$. In contrast to the Dirichlet problem we have no information about w along $\partial\Omega$. Therefore, the linear theory does not help us to prove the Hölder continuity of w which was used to obtain the Morrey estimate from (6.7). However, this step can be replaced by a different technical argument (see [37], Chapter 2, Theorem 7.2) which allows us to conclude the Morrey estimate

$$\int_{B_R(y_0)\cap D^+} |Dw|^2 \, dy \leq cR^{n-2+2\alpha} \qquad \text{for } w = D_k v, \ 1 \leq k \leq n-1 \tag{6.8}$$

based on (6.7). □

Remark 6.16. In order to establish the Hölder estimate for the gradient it suffices to have

$$a^i \in C^1(U_K), \quad a \in C^0(U_K), \quad \phi \in C^1(\overline{\Omega})$$

where $U_K := \overline{\Omega} \times [-K, K] \times [-K, K]^n$ and $K := \|u\|_{C^1(\overline{\Omega})}$.

7. The capillary surface problem

In this chapter we are interested in graphs of prescribed mean curvature that satisfy a contact angle condition at the boundary. That is the case for surfaces of liquids in capillary tubes. We call it the capillary surface problem (CSP) := $(\mathrm{CSP})_1$ where

$$(\mathrm{CSP})_\sigma \begin{cases} \operatorname{div}\left(\dfrac{Du}{\sqrt{1+|Du|^2}}\right) = \sigma H(\cdot, u, Du) & \text{in } \Omega \\[2ex] \dfrac{D_\mu u}{\sqrt{1+|Du|^2}} = \sigma\beta \qquad \text{on } \partial\Omega. \end{cases}$$

Here μ is the exterior unit normal of $\partial\Omega$ and β is the cosine of the contact angle.

Theorem 7.1 (Existence criterion for capillary surface problems). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{2,\alpha}$-boundary. Let $\beta \in C^{1,\alpha}(\overline{\Omega})$ and $H, H_z, H_{p^k} \in C^{0,\alpha}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$. Suppose that $H_z \geq \kappa > 0$. If the set*

$$\mathcal{M} := \left\{ u \in C^{2,\alpha}(\overline{\Omega}) \;\middle|\; \exists \sigma \in [0,1] \;:\; u \text{ is a solution of } (\mathrm{CSP})_\sigma \right\}$$

is bounded in $C^1(\overline{\Omega})$. Then the problem (CSP) *is solvable in $C^{2,\alpha}(\overline{\Omega})$.*

Proof. This follows directly from Theorem 6.15. Note that the σ-related problems are chosen slightly different here. □

Remark 7.2 (Notation). Recall our notation

$$a^i(p) := \frac{p^i}{\sqrt{1+|p|^2}}, \qquad a^{ij}(p) := \frac{\partial a^i(p)}{\partial p^j} = \frac{1}{\sqrt{1+|p|^2}}\left(\delta^{ij} - \frac{p^i p^j}{1+|p|^2}\right)$$

which we will use frequently in the following sections.

Remark 7.3 (Alternative approach for existence). One can also use a fixed point argument as for the Dirichlet problem to infer existence. However, due to the nonlinear boundary condition the corresponding operator has to be a map $T : C^{2,\alpha}(\overline{\Omega}) \to C^{2,\alpha}(\overline{\Omega}) : v \mapsto Tv$. One possible choice is to assign to v solutions $u = Tv$ of the problem

$$a^{ij}(Dv)D_{ij}u - u = H(\cdot, v, Dv) - v \quad \text{in } \Omega \qquad \frac{D_\mu u}{\sqrt{1+|Dv|^2}} = \beta \quad \text{on } \partial\Omega.$$

The artificial $-u$ is needed to assure unique solvability. The σ-related problems are as above, but the right hand side of the PDE is $\sigma H(\cdot, u, Du) + (1-\sigma)u$. Furthermore, one needs to establish a C^0-estimate for these related problems in order to prove compactness of T. This reduces the solvability to a priori estimates in $C^{2,\alpha}(\overline{\Omega})$. Finally, one can reduce to a priori estimates in $C^{1,\alpha}(\overline{\Omega})$ as in Theorem 6.13 and to a priori etimates in $C^1(\overline{\Omega})$ via Theorem 6.15. Note that the last two steps work under relatively mild assumptions on the coefficients, namely $a^i, a^i_{p^k} \in C^{1,\alpha}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$, $a \in C^{0,\alpha}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$, $\phi \in C^{1,\alpha}(\overline{\Omega})$. In this case the requirements for H will be needed to establish the C^0-estimate for the σ-related problems.

Most of the results in the following sections are due to Gerhardt [20]. Some improvements are also due to Huisken [31, 32].

7.1. C^0-estimate

Proposition 7.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{2,\alpha}$-boundary. Let $H \in C^1(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^{n+1})$, $H = H(x, z, \nu(p))$ with $H_z \geq \kappa > 0$ and $\beta \in C^1(\overline{\Omega})$ such that $|\beta| \leq 1 - a$ for some $a \in (0, 1]$. Then a solution $u \in C^{2,\alpha}(\overline{\Omega})$ of the problem* (CSP) *satisfies the estimate*

$$\sup_{\Omega} |u| \leq c\big(n, \kappa, a, \partial\Omega, \overline{H}\big), \qquad \overline{H} := \sup_{\Omega \times \{0\} \times [0,1]^{n+1}} |H|.$$

Proof. We want to apply the method of Stampacchia. We will need the following two inequalities. First the Sobolev inequality

$$\|u\|_{L^{n/(n-1)}(\Omega)} \leq c(n)\|Du\|_{L^1(\Omega)} + c(\partial\Omega)\|u\|_{L^1(\Omega)} \qquad \forall\ u \in W^{1,2}(\Omega) \tag{7.1}$$

and the inequality proven in [19], Lemma 1:

$$\int_{\partial\Omega} |u| \leq \int_{\Omega} |Du| + c(\partial\Omega) \int_{\Omega} |u| \qquad \forall\ u \in W^{1,2}(\Omega). \tag{7.2}$$

We multiply the PDE by the test function $u_k := \max\{u - k, 0\} \in W^{1,2}(\Omega)$ and integrate over Ω. Then we perform an integration by parts and take the boundary condition into account to obtain

$$0 = -\int_{\Omega} \Big(a^i(Du) D_i u_k + H(x, u, \nu(Du)) u_k\Big) + \int_{\partial\Omega} \beta u_k.$$

We use the notation $A(k) := \{x \in \Omega \mid u > k\}$ and estimate

$$\begin{aligned}
&\|Du_k\|_{L^1(\Omega)} - |A(k)| \le \int_{A(k)} \frac{|Du_k|^2}{\sqrt{1+|Du_k|^2}}\, dx = \int_{A(k)} a^i(Du_k) D_i u_k\, dx \\
&= -\int_{A(k)} H(\cdot, u_k, \nu(Du_k)) u_k\, dx + \int_{\partial\Omega} \beta u_k\, ds \\
&\le -\int_{A(k)} \int_0^1 \frac{d}{dt} H(\cdot, tu_k, \nu(Du_k)u) u_k\, dtdx - \int_{A(k)} H(\cdot, 0, \nu(Du_k)) u_k\, dx + |\beta| \int_{\partial\Omega} u_k\, ds \\
&\le -\int_{A(k)} \int_0^1 H_z(\cdot, tu_k, \nu(Du_k)) u_k^2\, dtdx + \overline{H} \int_{A(k)} u_k\, dx + |\beta| \int_{\partial\Omega} u_k\, ds \\
&\le -\kappa \int_{A(k)} u_k^2\, dx + \frac{\overline{H}}{2} \int_{A(k)} \left[\left(\frac{\kappa}{\overline{H}} \right) u_k^2 + \left(\frac{\overline{H}}{\kappa} \right) \right] dx + |\beta| \int_{\partial\Omega} u_k\, ds \\
&\le -\frac{\kappa}{2} \int_{A(k)} u_k^2\, dx + \frac{\overline{H}^2}{2\kappa} |A(k)| + |\beta| \int_{A(k)} \big(|Du_k| + c_1(\partial\Omega) |u_k| \big)\, dx \\
&\le -\frac{\kappa}{2} \int_{A(k)} u_k^2\, dx + \frac{\overline{H}^2}{2\kappa} |A(k)| + |\beta| \int_{A(k)} |Du_k|\, dx + \int_{A(k)} \left(2\frac{c_1^2|\beta|^2}{2\kappa} + \frac{1}{2} \frac{\kappa^2 |u_k|^2}{2} \right) dx \\
&\le -\frac{\kappa}{4} \int_{A(k)} u_k^2\, dx + \left(\frac{\overline{H}^2}{2\kappa} + \frac{c_1^2|\beta|^2}{\kappa^2} \right) |A(k)| + |\beta| \int_{A(k)} |Du_k|\, dx.
\end{aligned}$$

If $|\beta| \le 1 - a$ for $a \in (0,1)$ we obtain

$$a \int_\Omega |Du_k|\, dx + \frac{\kappa}{4} \int_\Omega u_k^2\, dx \le \left(1 + \frac{\overline{H}^2}{2\kappa} + \frac{c_1^2 a^2}{\kappa^2} \right) |A(k)|.$$

Applying the Sobolev and Hölder inequality we finally get

$$\begin{aligned}
&(h-k)|A(h)| \le \int_{A(h)} (u-k)\, dx \le \int_{A(k)} u_k\, dx \le |A(k)|^{1/n} \|u_k\|_{L^{n/(n-1)}(\Omega)} \\
&\le c_2(n, \Omega) |A(k)|^{1/n} \left(\int_\Omega |Du_k|\, dx + \int_\Omega u_k\, dx \right) \\
&\le \frac{c_2}{a} |A(k)|^{1/n} \left(a \int_\Omega |Du_k|\, dx + \frac{\kappa}{4} \int_\Omega u_k^2\, dx + \frac{1}{\kappa a^2} |A(k)| \right) \\
&\le \frac{c_2}{a} \left(1 + \frac{\overline{H}^2}{2\kappa} + \frac{c_1^2 a^2}{\kappa^2} + \frac{1}{\kappa a^2} \right) |A(k)|^{1+1/n}.
\end{aligned}$$

The Lemma of Stampacchia, Lemma 5.8, yields the upper bound. The lower bound can be obtained in a similar way. □

7.2. Global gradient estimate

To obtain the gradient estimate we use once more the method of Stampacchia. Let us start with a version of the Sobolev inequality.

Lemma 7.5 (**Michael-Simon-Sobolev Inequality**). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2-boundary. Let $u \in C^2(\overline{\Omega})$ and $M = \operatorname{graph} u$. Then we have*

$$\left(\int_M |z|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \le c(n)\int_M (|\nabla z| + |Hz|)\ d\mu + c(n)\int_{\partial\Omega} |z|\sqrt{1+|Du|^2}\ ds$$

for all $z \in C^1(\overline{\Omega})$. Note that ∇ stands for the tangential covariant derivative on M.

Proof. The inequality was first proved by Michael and Simon [46] in the case where z vanishes on $\partial\Omega$. The general case was proven by Gerhardt in [20], Lemma 1.1 by considering a sequence of functions $(z_k)_{k\in\mathbb{N}}$ having zero boundary values which converges to z. □

Lemma 7.6 (**Boundary integral estimate**). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2-boundary. Let $u \in C^2(\overline{\Omega})$ be a solution of* (CSP) *and $M = \operatorname{graph} u$. Then we have*

$$\int_{\partial\Omega} \left(\sqrt{1+|Du|^2} - \beta D_\mu u\right) z\, ds \le \int_M \Big(|\nabla\mu|z + |\nabla z| + |H|z\Big)\, d\mu$$

for all $z \in C^1(\overline{\Omega}, \mathbb{R}_{\ge 0})$. Again, ∇ stands for the tangential covariant derivative on M. The unit normal μ of $\partial\Omega$ was extended to be defined in all of $\overline{\Omega}$.

Proof. Let us put $W := \sqrt{1+|Du|^2}$ and denote that standard basis of $\mathbb{R}^{n+1}$ by $\{e_i\}_{1\le i\le n+1}$. Let $\phi \in C^1(\overline{\Omega})$. Using integration by parts we compute

$$\begin{aligned}
\int_M \langle \nabla\phi, e_i\rangle\ d\mu &= \int_\Omega \Big(D^i\phi - \langle \nu, D\phi\rangle \nu^i\Big) W\, dx \\
&= \int_\Omega \Big(D^i(\phi W) - \phi D^i W - a^k D_k(\phi D^i u) + a^k \phi D^i_k u\Big)\ dx \\
&= \int_{\partial\Omega} \phi W \mu^i\ ds + \int_\Omega D_k(a^k)\phi D^i u\ dx - \int_{\partial\Omega} \phi a^k D^i u \mu_k\ ds \\
&= \int_{\partial\Omega} \phi\Big(W\mu^i - \beta D^i u\Big)\ ds - \int_M H\phi\nu^i\, d\mu.
\end{aligned}$$

We observe that for $\phi_i := z\mu_i$ and summation over i we obtain

$$\int_{\partial\Omega} (W - \beta D_\mu u)\, z\, ds = \int_M \Big(\langle \nabla(z\mu_i), e_i\rangle + Hz\langle \mu, \nu\rangle\Big)\, d\mu$$

which yields the result. □

The major technical step in the argument of Stampacchia are the following estimates.

Proposition 7.7 (**Preparation of the Stampacchia argument**). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2-boundary. Suppose that $|\beta| \leq 1 - a$ for some $a \in (0,1]$ and $H_z \geq 0$. Let $u \in C^2(\overline{\Omega})$ be a solution of* (CSP) *and $M = \operatorname{graph} u$. Defining*

$$z := \max\{\ln v - k, 0\}, \qquad v := W - \beta D_\mu u, \qquad W := \sqrt{1 + |Du|^2}$$

we have

$$\int_{A(k)} \Big(|\nabla z|^2 + H^2 z\Big)\, d\mu \leq c|A(k)| \qquad \forall\ k \geq k_0 := 0$$

where $A(k) := \{(x, u(x)) \in M \mid \ln v(x) > k\}$ and $c = c(n, \beta, \partial\Omega, |H_x|, |H_z|)$.

Proof. We multiply the PDE from $D_i a^i = H$ by a test function $D^k \phi$ and integrate by parts w.r.t. D^k and D^i to obtain

$$\int_\Omega D_k a^i D_i \phi \, dx = -\int_\Omega \phi D_k H \, dx + \int_{\partial\Omega} \phi \mu_i D_k a^i \, ds \tag{7.3}$$

Choosing $\phi := (a^k - \beta\mu^k)\eta$ equality (7.3) reads

$$\begin{aligned}&\int_\Omega a^{ij} D_{jk} u \Big(\eta D_i a^k + a^k D_i \eta - D_i(\beta\mu^k)\eta - \beta\mu^k D_i \eta\Big)\, dx \\ &\qquad = -\int_\Omega (a^k - \beta\mu^k)\eta D_k H \, dx + \int_{\partial\Omega} (a^k - \beta\mu^k)\eta\mu_i a^{ij} D_{jk} u \, ds. \end{aligned}\tag{7.4}$$

Note that $D_j v = (a^k - \beta\mu^k) D_{kj} u - D_j(\beta\mu^k) D_k u$. Plugging this into (7.4) we get

$$\begin{aligned}&\int_\Omega a^{ij} D_i \eta \Big(D_j v + D_j(\beta\mu^k) D_k u\Big)\, dx \\ &\qquad = -\int_\Omega \Big[a^{ij} D_{jk} u \Big(a^{kl} D_{il} u - D_i(\beta\mu^k)\Big) + (a^k - \beta\mu^k) D_k H\Big]\eta \, dx \\ &\qquad\quad + \int_{\partial\Omega} \eta\mu_i a^{ij} \Big(D_j v + D_j(\beta\mu^k) D_k u\Big)\, ds. \end{aligned}\tag{7.5}$$

To proceed further we choose $\eta := zv$ and make the following claims:

Claim 1:

$$a^2 \int_{A(k)} |\nabla z|^2 \, d\mu \leq \int_\Omega a^{ij} D_i(vz) D_j v \, dx.$$

Claim 2:

$$-\int_\Omega a^{ij} D_i(vz) D_j(\beta\mu^k) D_k u \, dx \leq c \int_{A(k)} (1 + z)\, d\mu + \frac{a^2}{3} \int_{A(k)} |\nabla z|^2 \, d\mu.$$

Claim 3:

$$-a^{ij} D_{jk} u \Big(a^{kl} D_{il} u - D_i(\beta\mu^k)\Big) \leq c - \frac{1}{2}|A|^2.$$

Claim 4:

$$\int_{\partial\Omega} \mu_i a^{ij}\Big(D_j v + D_j(\beta\mu_k)D^k u\Big)\eta\, ds \le c\int_{\partial\Omega}\eta\, ds.$$

Using these claims in (7.5) we obtain

$$\begin{aligned} a^2\int_{A(k)}|\nabla z|^2\, d\mu &\le c\int_{A(k)}(1+z)\ d\mu + \frac{a^2}{3}\int_{A(k)}|\nabla z|^2\, d\mu \\ &+ \int_{\Omega}\Big(c - \frac{1}{2}|A|^2 - (a^k - \beta\mu^k)(H_{x^k} + H_z D_k u)\Big)\, zv\, dx + c\int_{\partial\Omega} zv\, ds. \end{aligned} \tag{7.6}$$

From $|\beta| \le 1-a$ we can deduce that $aW \le v \le 2W$. Furthermore, we use Lemma 7.6 to estimate the boundary integral and recall that $H_z \ge 0$. We get

$$\begin{aligned} \frac{2a^2}{3}\int_{A(k)}|\nabla z|^2\, d\mu &\le c\int_{A(k)}(1+z)\ d\mu + \int_{A(k)}\Big(2c - \frac{a}{2}|A|^2 + 4|H_x| + 2\frac{\beta^2}{W}H_z\Big)\, z\, d\mu \\ &\qquad + c\int_{A(k)}\Big(|H|z + |\nabla\mu|z + |\nabla z|\Big)\, d\mu \\ &\le c\int_{A(k)}(1+z+|\nabla z|)\, d\mu + \int_{A(k)}\Big(c|H| - \frac{a}{2n}H^2\Big)\, z\, d\mu \\ &\le c\int_{A(k)}(1+z)\, d\mu + \frac{a^2}{3}\int_{A(k)}|\nabla z|^2\, d\mu - \int_{A(k)}\frac{|H|^2}{3}z\, d\mu. \end{aligned} \tag{7.7}$$

where we used Young's inequality on the $|\nabla z|$-term and on the $|H|$-term together with the fact that $|A|^2 \ge H^2/n$. To finish the proof it suffices to verify that

$$c\int_{A(k)} z\, d\mu \le \frac{1}{2}\int_{A(k)}\Big(|\nabla z|^2 + H^2 z\Big)\ d\mu + c|A(k)| \tag{7.8}$$

holds. We start with the weak formulation with test function $\phi = zu$

$$\begin{aligned} 0 = -\int_{\Omega}\Big(D_i a^i - H\Big)\, zu\, dx &= \int_{\Omega}\Big(za^i D_i u + ua^i D_i z + Hzu\Big)\ dx - \int_{\partial\Omega} a^i\mu_i zu\, ds \\ &= \int_{\Omega}\Big(zW - \frac{z}{W} + ua^i D_i z + Huz\Big)\ dx - \int_{\partial\Omega}\beta zu\, ds. \end{aligned}$$

Note that for $k \ge 0$ we have $z \le W$. Using (7.2) to convert the boundary term into a volume term we get

$$\begin{aligned}
\int_{A(k)} z\,d\mu &\le \int_\Omega \left(\frac{z}{W} + |u||Dz| + |Hu|z\right) dx + c\int_{\partial\Omega} z\,ds \\
&\le c\int_{A(k)} (z + |Dz| + |H|z)\, W^{-1}\,d\mu \\
&\le c\int_{A(k)} \left(W + \frac{|Dz|}{\sqrt{2cW}}\sqrt{2cW} + \left(\frac{|H|\sqrt{z}}{\sqrt{2c}}\right)\sqrt{2cz}\right) W^{-1}\,d\mu \\
&\le \int_{A(k)} \left(cW + \frac{|Dz|^2}{2W} + \frac{|H|^2 z}{2}\right) W^{-1}\,d\mu \\
&\le \int_{A(k)} \left(c + \frac{|\nabla z|^2}{2} + \frac{|H|^2 z}{2}\right) d\mu.
\end{aligned}$$

In the last step we used that $|Dz| \le W|\nabla z|$. □

Before we come the the actual gradient estimate let us verify the claims we made during the proof of Proposition 7.7.

Exercise II.8. Let $M = \operatorname{graph} u$ where $u : \overline{\Omega} \to R$. Let $f, g \in C^1(\overline{\Omega})$, $W(Du) := \sqrt{1 + |Du|^2}$ and $a^{ij}(Du) := \partial a^i/\partial p^j|_{Du}$ where $a^i(p) := p^i/W(p)$. Show that the following relations hold

$$W a^{ij}(Du) D_i f D_j f = |\nabla f|^2, \qquad a^{ij}(Du) D_i f D_j g \le W^{-1}(Du)|\nabla f||Dg|$$

where $\nabla f := Df - \langle Df, \nu\rangle\nu$.

Lemma 7.8 (Claim 1). *Using the same assumptions as in Proposition 7.7 we obtain the following estimate*

$$a^2 \int_{A(k)} |\nabla z|^2\,d\mu \le \int_\Omega a^{ij} D_i(vz) D_j v\,dx.$$

Proof. On $A(k)$ we have $D_i z = v^{-1} D_i v$. Futhermore, we note that for any $f \in C^1(\overline{\Omega})$ we have $W a^{ij} D_i f D_j f = |\nabla f|^2$. Using these equalities we compute

$$\begin{aligned}
\int_\Omega a^{ij} D_i(vz) D_j v\,dx &= \int_{A(k)} \Big(z a^{ij} D_i v D_j v + v a^{ij} D_i z D_j v\Big) W^{-1}\,d\mu \\
&= \int_{A(k)} v^2 a^{ij} D_i z D_j z (1+z) W^{-1}\,d\mu \ge \int_{A(k)} \frac{v^2}{W^2}|\nabla z|^2\,d\mu \ge a^2 \int_{A(k)} |\nabla z|^2\,d\mu
\end{aligned}$$

since z is assumed to be positive and $aW \le v$. □

Lemma 7.9 (Claim 2). *Using the same assumptions as in Proposition 7.7 we obtain the following estimate*

$$-\int_\Omega a^{ij} D_j(\beta\mu^k) D_k u D_i(vz)\,dx \le c\int_{A(k)} (1+z)\,d\mu + \frac{a}{4}\int_{A(k)} |\nabla z|^2\,d\mu$$

where $c = c(a, |D(\beta\mu)|)$.

Proof. We note that for $f, g \in C^1(\overline{\Omega})$ we have $a^{ij} D_i f D_j g \le W^{-1}|\nabla f||Dg|$ and compute

$$\begin{aligned}
-\int_\Omega a^{ij} D_j(\beta\mu^k) D_k u D_i(vz)\,dx &= -\int_\Omega a^{ij} D_j(\beta\mu^k) D_k u (zD_i v + vD_i z)\,dx \\
&\le \int_\Omega |\nabla z||D(\beta\mu)|W^{-1}|Du|(z+1)v\,dx \le 2c\int_{A(k)} |\nabla z|(z+1)\,d\mu \\
&\le c\int_{A(k)} \left(\frac{a^2}{4c}|\nabla z|^2 + \frac{4c}{a^2}(z+1)^2\right) d\mu \le \frac{a^2}{4}\int_{A(k)} |\nabla z|^2\,d\mu + c\int_{A(k)} (1+z)\,d\mu.
\end{aligned}$$

In the last step we used the fact that $(1+z)^2 \le 2(1+\ln v + (\ln v)^2) \le 6(1+\ln v)$. □

Lemma 7.10 (Claim 3). *Using the same assumptions as in Proposition 7.7 we obtain the following estimate*

$$a^{ij} D_{jk} u\Big(a^{kl} D_{il} u - D_i(\beta\mu^k)\Big) \ge \frac{|A|^2}{2} - c$$

where $c = c(|D(\beta\mu)|)$. In the case $\beta \equiv 0$ we have $\big(a^{ij} D_{jk} u\big)\big(a^{kl} D_{il} u\big) = |A|^2$.

Proof. The coefficients of the metric of $M = \operatorname{graph} u$ w.r.t. the usual tangent space basis are $g_{ij} = \delta_{ij} + D_i u D_j u$. The inverse metric and the second fundamental form have the coefficients

$$g^{ij} = \left(\delta^{ij} - \frac{D^i u D^j u}{1+|Du|^2}\right), \qquad h_{ij} = \frac{D_{ij} u}{\sqrt{1+|Du|^2}}.$$

Therefore, we see that

$$|A|^2 = A_i^k A_k^i = \Big[g^{-1}h\Big]_i^k \Big[g^{-1}h\Big]_k^i = \big(a^{ij} D_{jk} u\big)\big(a^{kl} D_{il} u\big)$$

and thus

$$a^{ij} D_{jk} u\Big(a^{kl} D_{il} u - D_i(\beta\mu^k)\Big) \ge |A|^2 - c|A| \ge \frac{|A|^2}{2} - c$$

where we used Young's inequality in the last step. □

Lemma 7.11 (Claim 4). *Using the same assumptions as in Proposition 7.7 we obtain the following estimate*

$$\int_{\partial\Omega} \eta\mu_i a^{ij}\Big(D_j v + D_j(\beta\mu^k) D_k u\Big)\,ds \le c\int_{\partial\Omega} \eta\,ds$$

where $c = c(|\beta|, |D\beta|, |D\mu|)$.

Proof. Let $x_0 \in \partial\Omega$. We choose a coordinate system $\{y^k\}_{k\in\{1,\dots,n\}}$ centered at x_0 such that $y^n = -\mu$. Thus, $\partial y^n/\partial x^k = -\mu^k$ and by the Neumann condition

$$(a^k - \beta\mu^k)\frac{\partial y^n}{\partial x^k} = 0$$

Since $y^i \in T_{x_0}\partial\Omega$ for $i = 1, ..., n-1$ we can apply the following operator to the Neumann condition to get zero on the left hand side

$$\begin{aligned}
0 &= (a^k - \beta\mu^k)\frac{\partial y^r}{\partial x^k}\frac{\partial}{\partial y^r}\left(a^i\mu_i - \beta\right)\\
&= (a^k - \beta\mu^k)\frac{\partial y^r}{\partial x^k}\left(\mu_i a^{ij}\frac{\partial D_j u}{\partial x^s}\frac{\partial x^s}{\partial y^r} + a^i\frac{\partial \mu_i}{\partial x^p}\frac{\partial x^p}{\partial y^r} - \frac{\partial \beta}{\partial x^p}\frac{\partial x^p}{\partial y^r}\right)\\
&= (a^k - \beta\mu^k)\left(\mu_i a^{ij} D_{jk}u + a^i D_k\mu_i - D_k\beta\right)\\
&= a^{ij}\mu_i\Big(D_j v + D_j(\beta\mu^l)D_l u\Big) + (a^k - \beta\mu^k)\Big(a^i D_k\mu_i - D_k\beta\Big).
\end{aligned}$$

Thus, the boundary integral can be estimated as follows

$$\begin{aligned}
&\int_{\partial\Omega}\eta\mu_i a^{ij}\Big(D_j v + D_j(\beta\mu^k)D_k u\Big)\,ds\\
&\qquad = \int_{\partial\Omega}\eta(a^k - \beta\mu^k)\Big(D_k\beta - a^i D_k\mu_i\Big)\,ds \le c\int_{\partial\Omega}\eta\,ds
\end{aligned}$$

with $c = c(|\beta|, |D\beta|, |D\mu|)$. □

Finally, we can use the method of Stampacchia to conclude the gradient estimate.

Theorem 7.12 (Gradient estimate for the capillary surface problem). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2-boundary. Let $H \in C^1(\overline{\Omega}\times\mathbb{R})$ and $\beta \in C^1(\overline{\Omega})$. Suppose that $|\beta| \le 1 - a$ for some $a \in (0, 1]$ and that $H_z \ge \kappa > 0$. Then a solution $u \in C^{2,\alpha}(\overline{\Omega})$ of* (CSP) *is a priori bounded in $C^1(\overline{\Omega})$.*

Proof. The $|u|$ estimate was already derived in Proposition 7.4. To derive the gradient estimate we combine the Michael-Simon-Sobolev inequality from Lemma 7.5, the transformation of the boundary integral Lemma 7.6 and the estimate from Proposition 7.7 to obtain

$$\begin{aligned}
(h-k)|A(h)| &\le \int_{A(k)} z\,d\mu\\
&\le \left(\int_{A(k)}|z|^{\frac{n}{n-1}}\,d\mu\right)^{\frac{n-1}{n}}|A(k)|^{1/n}\\
&\le c\left[\int_{A(k)}\Big(|\nabla z| + |H|z\Big)\,d\mu + \int_{\partial\Omega} zW\,ds\right]|A(k)|^{1/n}\\
&\le c\left[\int_{A(k)}\Big(|\nabla z| + |H|z\Big)\,d\mu + \frac{c}{a}\int_{A(k)}\Big(|\nabla z| + |H|z + z\Big)\,d\mu\right]|A(k)|^{1/n}\\
&\le c\int_{A(k)}\Big(|\nabla z|^2 + H^2 z + z\Big)\,d\mu\,|A(k)|^{1/n}\\
&\le c|A(k)|^{1+1/n}.
\end{aligned}$$

To estimate the integral of z in the last step we used (7.8) once more. Finally, the Lemma of Stampacchia implies an estimate for $\ln v$

$$\ln v \leq k_0 + c|A(k_0)|^{1/n} \leq c|M|^{1/n}.$$

Since $a|Du| \leq v$ it is left to show that $|M|$ is bounded. We use the weak formulation with test function $\phi = u$ to compute

$$0 = \int\limits_{\Omega} \Big(a^i D_i u + Hu\Big)\, dx - \int\limits_{\partial\Omega} a^i u \mu_i \, ds = \int\limits_{\Omega} \Big(W - \frac{1}{W} + Hu\Big)\, dx - \int\limits_{\partial\Omega} \beta u \, ds.$$

Thus

$$|M| = \int_{\Omega} W\, dx \leq \int_{\Omega} \Big(\frac{1}{W} + |Hu|\Big)\, dx + \int_{\partial\Omega} \beta u \, ds \leq |\Omega|(1 + |uH|) + |\partial\Omega||u|$$

is bounded since $|u| \leq c$ is already known. □

7.3. Existence and uniqueness theorem

Theorem 7.13 (**Existence for capillary surface problems**). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{2,\alpha}$-boundary. Let $H \in C^{1,\alpha}(\overline{\Omega} \times \mathbb{R})$ with $H_z \geq \kappa > 0$ and $\beta \in C^{1,\alpha}(\overline{\Omega})$ such that $|\beta| \leq 1 - a$ for some $a \in (0,1]$. Then the capillary surface problem* (CSP) *has aunique solution $u \in C^{2,\alpha}(\overline{\Omega})$.*

Proof. The result is based on Theorem 7.1. The necessary a priori estimates are the sup-estimate from Proposition 7.4 and the gradient estimates from Theorem 7.12. □

Remark 7.14 (**More general right hand sides**). Note that the sup-estimate only requires H to be bounded in the sense that

$$\sup_{(x,p)\in\Omega\times\mathbb{R}^n} |H(x,0,p)| \leq c < \infty. \tag{7.9}$$

Furthermore, the interior gradient estimate works as for the prescribed mean curvature problem. Therefore, given H as above but with an additional dependence on $|Du|$ existence is guaranteed as long as (7.9) holds and and $|Du|$ is controlled near $\partial\Omega$. The latter might be achieved by a modification of the Stampacchia argument or even a totally different approach.

Remark 7.15. There are many interesting related problems which can be treated by similar techniques. Such as the capillary surface problem involving an obstacle (see Huisken: [32]). This leads to the study of variational inequalities instead of a PDE. Another family of problems would be to replace the curvature which is prescribed. One could for example look at prescribed Gauss curvature. There the PDE is fully non-linear and C^2 a priori estimates are required to obtain an existence result.

Part III.

Geometric evolution equations

8. Classical solutions of MCF and IMCF

8.1. Short-time existence

Let us first consider nonlinear parabolic Dirichlet problems with initial value u_0, i.e.

$$(\mathrm{NP})_D \begin{cases} \dfrac{\partial u}{\partial t} - Q(\cdot, u, Du, D^2u) = 0 & \text{in } \Omega \times (0,T) \\ u = \phi & \text{on } \partial\Omega \times (0,T) \\ u = u_0 & \text{on } \Omega \times \{0\}. \end{cases}$$

We will use the following notation:

Definition 8.1. Using the shortcuts

$$Q_{q^{ij}}|_v := \frac{\partial Q(\cdot, z, p, q)}{\partial q^{ij}}\Big|_{(\cdot, v, Dv, D^2v)}, \qquad Q_{p^k}|_v := \frac{\partial Q(\cdot, z, p, q)}{\partial p^k}\Big|_{(\cdot, v, Dv, D^2v)},$$

$$Q_z|_v := \frac{\partial Q(\cdot, z, p, q)}{\partial z}\Big|_{(\cdot, v, Dv, D^2v)}, \qquad Q[u] := Q(\cdot, u, Du, D^2u).$$

we define the linearization of $\frac{\partial}{\partial t} - Q$ around some $v \in C^{2,\alpha;1,\frac{\alpha}{2}}(\overline{Q_T})$ by

$$\frac{\partial}{\partial t} - L(Q,v) := \frac{\partial}{\partial t} - \left(Q_{q^{ij}}|_v D_{ij} + Q_{p^k}|_v D_k + Q_z|_v \right).$$

We obtain the following existence criterion:

Theorem 8.2 (**Short-time existence for nonlinear Dirichlet problems**). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{2,\alpha}$-boundary and $T > 0$. Let $\phi \in C^{2,\alpha;1,\frac{\alpha}{2}}(\overline{Q_T})$ and $u_0 \in C^{2,\alpha}(\overline{\Omega})$. Suppose that $0 < \beta \le \alpha$ and Q has the following regularity*

$$Q, Q_z, Q_{p^k}, Q_{q^{ij}} \in C^{0,\beta;0,\frac{\beta}{2}} \quad \text{w.r.t. } (x,t) \in \overline{Q}_T$$

$$Q_z, Q_{p^k}, Q_{q^{ij}} \in C^{0,\beta} \quad \text{w.r.t. } (z,p) \in \mathbb{R} \times \mathbb{R}^n$$

$$Q_z, Q_{p^k}, Q_{q^{ij}} \in C^{0,1} \quad \text{w.r.t. } q \in \mathbb{R}^{n\times n}$$

and that the compatibility conditions of first order are satisfied, i.e.

$$\phi(\cdot, 0) = u_0\big|_{\partial\Omega}, \qquad \frac{\partial \phi}{\partial t}\Big|_{(\cdot,0)} = Q(\cdot, u_0, Du_0, D^2u_0)\big|_{\partial\Omega}.$$

If there exists some $v \in C^{2,\alpha;1,\frac{\alpha}{2}}(\overline{Q_T})$ such that

1. *$\frac{\partial}{\partial t} - L(Q,v)$ is uniformly parabolic*

2. $v = \phi$ *on* $\partial\Omega \times (0, T)$*,*

3. $v = u_0$ *on* $\Omega \times \{0\}$*.*

Then there exists some $\varepsilon \in (0, T)$ *and a solution* $u \in C^{2,\beta;1,\frac{\beta}{2}}(\overline{Q_\varepsilon})$ *of* (NP).

Remark 8.3 (Regularity issue). Note that in general we can not choose $\beta = \alpha$ even if the first condition is satisfied with α. The reason is the convergence of the norm in the last line of the proof. However, $\beta = \alpha$ might be possible if one has further information on Q (cf. short-time existence for mean curvature flow below).

Remark 8.4 (A possible choice of v). One possibility is to choose v as the solution of

$$\begin{cases} \dfrac{\partial v}{\partial t} - \Delta v = Q[u_0] - \Delta u_0 & \text{in } \Omega \times (0, T) \\ v = \phi & \text{on } \partial\Omega \times (0, T) \\ v = u_0 & \text{on } \Omega \times \{0\}. \end{cases}$$

It that case it only remains to check the first condition, i.e. uniform ellipticity of $[Q_{q^{ij}}|_v]$.

Proof of the Theorem. Let us first reduce to the case of zero initial data by defining $\tilde{u} := u - u_0$. Then $\tilde{u}$ satisfies

$$\widetilde{(\mathrm{NP})}_D \begin{cases} \dfrac{\partial \tilde{u}}{\partial t} - Q[\tilde{u} + u_0] = 0 & \text{in } \Omega \times (0, T) \\ \tilde{u} = \phi - u_0 & \text{on } \partial\Omega \times (0, T) \\ \tilde{u} = 0 & \text{on } \Omega \times \{0\}. \end{cases}$$

The existence of solutions of $\widetilde{(\mathrm{NP})}_D$ is equivalent to the invertibility of the operator

$$S : X_T \to Y_T : w \mapsto Sw := (S_1 w, S_2 w) := \left(\frac{\partial w}{\partial t} - Q[w + u_0], (w + u_0)\big|_{S_T} - \phi \right)$$

around $(0, 0) \in Y$ where the spaces X_T and Y_T are defined as

$$X_T := \left\{ w \in C^{2,\beta;1,\frac{\beta}{2}}(\overline{Q_T}) \;\middle|\; w(\cdot, 0) = 0 \right\},$$

$$Y_T := \left\{ (f, \psi) \in C^{0,\beta;0,\frac{\beta}{2}}(\overline{Q_T}) \times C^{2,\beta;1,\frac{\beta}{2}}(\overline{S_T}) \;\middle|\; \psi(\cdot, 0) = 0, \quad \frac{\partial \psi}{\partial t}\Big|_{(\cdot,0)} = f\big|_{S_T}(\cdot, 0) \right\}.$$

We endow X_T with its usual Hölder norm and put on Y_T the sum of the Hölder norms of the two factors. Note that due to the compatibility conditions S maps indeed into Y_T since

$$(S_2 w)(\cdot, 0) = \left((w + u_0)\big|_{\partial\Omega} - \phi \right)(\cdot, 0) = u_0\big|_{\partial\Omega} - \phi(\cdot, 0) = 0$$

and

$$\frac{\partial (S_2 w)}{\partial t}\Big|_{(\cdot,0)} - (S_1 w)\big|_{S_T}(\cdot, 0) = -\frac{\partial \phi}{\partial t}\Big|_{(\cdot,0)} + Q[u_0]\Big|_{S_T}(\cdot, 0) = 0.$$

Next, we observe that S is continuously Fréchet-differentiable in a neighborhood of $\tilde{v} := v - u_0 \in X$ with derivative

$$DS\big|_{\tilde{v}} : X_T \to Y_T : w \mapsto (DS\big|_{\tilde{v}})(w) = \left(\frac{\partial w}{\partial t} - L(Q, \tilde{v} + u_0)w, w\big|_{S_T}\right)$$
$$= \left(\frac{\partial w}{\partial t} - L(Q, v)w, w\big|_{S_T}\right).$$

By assumption $L(Q, v)$ is uniformly parabolic with coefficients in $C^{0,\beta;0,\frac{\beta}{2}}(\overline{Q_T})$. Furthermore, for all $(f, \psi) \in Y_T$ the linear problem

$$\text{(LP)} \begin{cases} \dfrac{\partial w}{\partial t} - L(Q, v)w = f & \text{in } \Omega \times (0, T) \\ w = \psi & \text{on } \partial\Omega \times (0, T) \\ w = 0 =: w_0 & \text{on } \Omega \times \{0\}. \end{cases}$$

satisfies the first order compatibility conditions, i.e. on $\partial\Omega \times (0, T)$ we have

$$\psi(\cdot, 0) = 0 = w_0, \qquad \frac{\partial \psi}{\partial t}\Big|_{(\cdot,0)} = f(\cdot, 0) = L(Q, v(\cdot, 0))w_0 + f(\cdot, 0).$$

Therefore, by the existence theorem for linear parabolic Dirichlet problems, Theorem 3.26, there exists a unique solution of (LP), i.e. $DS\big|_{\tilde{v}}$ is a linear homeomorphism from X to Y. Thus, the inverse function theorem implies the invertibility of S in a neighborhood of $S\tilde{v}$. It remains to show that for some $\varepsilon > 0$ the point $(0, 0)$ is arbitrary close to $S\tilde{v}$ in the norm of Y_ε, i.e.

$$\|S\tilde{v}\|_{Y_\varepsilon} = \left\|\left(\frac{\partial \tilde{v}}{\partial t} - Q[\tilde{v} + u_0], 0\right)\right\|_{Y_\varepsilon} = \left\|\frac{\partial v}{\partial t} - Q[v]\right\|_{C^{0,\beta;0,\frac{\beta}{2}}(\overline{Q_\varepsilon})} \to 0$$

as $\varepsilon \to 0$. This is true since the function $\frac{\partial v}{\partial t} - Q[v]$ vanishes at $t = 0$ (since $\frac{\partial v}{\partial t}|_0 = \frac{\partial \phi}{\partial t}|_0 = Q[u_0] = Q[v_0]$) and has a bit more regularity (α instead of β) than required in the norm above. □

Remark 8.5 (Compatibility conditions). In some cases it might be desirable to consider Dirichlet problems where $\phi(\cdot, 0) = u_0$ on $\partial\Omega$ but where the first order compatibility condition is not satisfied. In that case one can not expect to obtain a solution $u \in C^{2,\alpha,1,\frac{\alpha}{2}}(\overline{Q_T})$. So one has to replace these Hölder spaces by weighted Hölder spaces which contain functions whose derivatives have discontinuities as they approach $(\partial\Omega \times [0, T]) \cap (\overline{\Omega} \times \{0\}) = \partial\Omega \times \{0\}$. We will come back to this discussion when we talk about mean curvature flow in the next section.

Let us now consider nonlinear parabolic Neumann problems with initial value u_0, i.e.

$$\text{(NP)}_N \begin{cases} \dfrac{\partial u}{\partial t} - Q(\cdot, u, Du, D^2u) = 0 & \text{in } \Omega \times (0, T) \\ G(\cdot, u, Du) = 0 & \text{on } \partial\Omega \times (0, T) \\ u = u_0 & \text{on } \Omega \times \{0\}. \end{cases}$$

We denote the outward unit normal to $S_T := \partial\Omega \times (0,T)$ by μ and the linearization of G around some v by N, i.e.

$$N(G,v)w := G_{p^k}|_v D_k w + G_z|_v w := \frac{\partial G}{\partial p^k}\Big|_{(\cdot,v,Dv)} D_k w + \frac{\partial G}{\partial z}\Big|_{(\cdot,v,Dv)} w.$$

Let us assume that the transversality condition

$$\left\langle \frac{\partial G}{\partial p}, \mu \right\rangle = 0 \qquad \text{on } S_T$$

is satisfied. Then we obtain a similar existence criterion:

Theorem 8.6 (Short-time existence for nonlinear Neumann problems). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{2,\alpha}$-boundary and $T > 0$. Let $\phi \in C^{1,\alpha;0,\frac{\alpha}{2}}(\overline{Q_T})$ and $u_0 \in C^{2,\alpha}(\overline{\Omega})$. Suppose that Q has the same regularity as in Theorem 8.2 and that G has the following regularity.*

$$G, G_z, G_{p^k} \in C^{1,\beta;0,\frac{\beta}{2}} \quad \text{w.r.t. } (x,t) \in \partial\Omega \times [0,T]$$

$$G_z, G_{p^k} \in C^{1,\beta} \qquad \text{w.r.t. } (z,p) \in \mathbb{R} \times \mathbb{R}^n.$$

In addition we assume that the zero order compatibility condition is satisfied, i.e.

$$G(\cdot, 0, u_0, Du_0) = 0 \qquad \text{on } \partial\Omega.$$

If there exists some $v \in C^{2,\alpha;1,\frac{\alpha}{2}}(\overline{Q_T})$ and some $\beta \in (0,\alpha)$ such that

1. *$\frac{\partial}{\partial t} - L(Q,v)$ is uniformly parabolic*
2. *$\frac{\partial v}{\partial t}\Big|_{t=0} = Q[u_0]$ in $\overline{\Omega}$,*
3. *$v = u_0$ on $\Omega \times \{0\}$.*

Then there exists some $\varepsilon \in (0,T)$ and a solution $u \in C^{2,\beta;1,\frac{\beta}{2}}(\overline{Q_\varepsilon})$ of $(\mathrm{NP})_N$.

Remark 8.7 (Regularity issue). Note that in general we can not choose $\beta = \alpha$ even if the regularity assumptions on F and G are satisfied with α instead of β. The reason is the convergence of the norm in the last line of the proof. However, $\beta = \alpha$ might be possible if one has further information on Q and G.

Remark 8.8 (A possible choice of v). One possibility is to choose v as the solution of

$$\begin{cases} \dfrac{\partial v}{\partial t} - \Delta v = Q[u_0] - \Delta u_0 & \text{in } \Omega \times (0,T) \\ \mu^k D_k v = D_\mu u_0 & \text{on } \partial\Omega \times (0,T) \\ v = u_0 & \text{on } \Omega \times \{0\}. \end{cases}$$

It that case it only remains to check the first condition, i.e. uniform ellitpcity of $[Q_{q^{ij}}|_v]$.

Proof of the Theorem. Let us first reduce to the case of zero initial data by defining $\tilde{u} := u - u_0$. Then $\tilde{u}$ satisfies

$$\widetilde{(\mathrm{NP})}_N \begin{cases} \dfrac{\partial \tilde{u}}{\partial t} - Q[\tilde{u} + u_0] = 0 & \text{in } \Omega \times (0,T) \\ G[\tilde{u} + u_0] = 0 & \text{on } \partial\Omega \times (0,T) \\ \tilde{u} = 0 & \text{on } \Omega \times \{0\}. \end{cases}$$

The existence of solutions of $\widetilde{(\mathrm{NP})}_N$ is equivalent to the invertibility of the operator

$$S : X \to Y : w \mapsto Sw := (S_1 w, S_2 w) := \left(\frac{\partial w}{\partial t} - Q[w + u_0], G[w + u_0] \right)$$

around $(0,0) \in Y$ where the spaces X and Y are defined as

$$X_T := \left\{ w \in C^{2,\beta;1,\frac{\beta}{2}}(\overline{Q_T}) \;\middle|\; w(\cdot, 0) = 0 \right\},$$

$$Y_T := \left\{ (f, \psi) \in C^{0,\beta;0,\frac{\beta}{2}}(\overline{Q_T}) \times C^{1,\beta;0,\frac{\beta}{2}}(\overline{S_T}) \;\middle|\; \psi(\cdot, 0) = 0 \right\}.$$

We endow X_T with its usual Hölder norm and put on Y_T the sum of the Hölder norms of the two factors. Note that due to the compatibility conditions S maps indeed into Y_T as

$$(S_2 w)(\cdot, 0) = G[w(\cdot, 0) + u_0] = G(\cdot, 0, u_0, Du_0) = 0.$$

Next, we observe that S is continuously Fréchet-differentiable in a neighborhood of $\tilde{v} := v - u_0 \in X$ with derivative

$$DS\big|_{\tilde{v}} : X_T \to Y_T : w \mapsto (DS\big|_{\tilde{v}})(w) = \left(\frac{\partial w}{\partial t} - L(Q, \tilde{v} + u_0)w, N(G, \tilde{v} + u_0)w \right)$$
$$= \left(\frac{\partial w}{\partial t} - L(Q, v)w, N(G, v)w \right).$$

By assumption $\frac{\partial}{\partial t} - L(Q, v)$ is uniformly parabolic with coefficients in $C^{0,\beta;0,\frac{\beta}{2}}(\overline{Q_T})$. Furthermore, $N(G, v)$ satisfies the transversality conditon and has coefficients in $C^{1,\beta;0,\frac{\beta}{2}}(\overline{S_T})$. Finally, for all $(f, \psi) \in Y_T$ the linear problem

$$(\mathrm{LP}) \begin{cases} L(Q, v)w = f & \text{in } \Omega \times (0,T) \\ N(G, v)w = \psi & \text{on } \partial\Omega \times (0,T) \\ w = 0 =: w_0 & \text{on } \Omega \times \{0\}. \end{cases}$$

satisfies the zero order compatibility conditions, i.e. we have

$$\psi(\cdot, 0) = 0 = N(G, v_0)w_0 \qquad \text{on } S_T.$$

Therefore, by the existence theorem for linear parabolic Neumann problems, Theorem 3.27, there exists a unique solution of (LP), i.e. $DS\big|_{\tilde{v}}$ is a linear homeomorphism from X_T to Y_T. Thus, the inverse function theorem implies the invertibility of S in a neighborhood of $S\tilde{v}$. It remains to show that for some $\varepsilon > 0$ the point $(0,0)$ is arbitrary close to $S\tilde{v}$ in the norm of Y_ε, i.e.

$$\begin{aligned}\|S\tilde{v}\|_{Y_\varepsilon} &= \left\| \left(\frac{\partial \tilde{v}}{\partial t} - Q[\tilde{v} + u_0], G[\tilde{v} + u_0] \right) \right\|_{Y_\varepsilon} \\ &= \left\| \frac{\partial v}{\partial t} - Q[v] \right\|_{C^{0,\beta;0,\frac{\beta}{2}}(\overline{Q_\varepsilon})} + \|G(\cdot, v, Dv)\|_{C^{1,\beta;0,\frac{\beta}{2}}(\overline{S_\varepsilon})} \to 0\end{aligned}$$

as $\varepsilon \to 0$. This is true since the functions $\frac{\partial v}{\partial t} - Q[v]$ and $G(\cdot, v, Dv)$ vanish at $t = 0$ (recall that $v(\cdot, 0) = u_0$, $\frac{\partial v}{\partial t}|_0 = Q[u_0]$, $G[u_0] = 0$) and have a bit more regularity (α instead of β) than required by these norms. □

8.2. Evolving graphs under mean curvature flow

First, we want to consider graphs over a domain $\Omega \subset \mathbb{R}^n$ which move in the direction of the unit normal with speed equal to some scalar function f. We assume that the initial hypersurface is given by

$$F_0 : \overline{\Omega} \to \mathbb{R}^{n+1} : x \mapsto F_0(x) := (x, u_0(x))$$

for some given $u_0 \in C^{2,\alpha}(\overline{\Omega})$ and that the boundary is kept fix[1]. So our goal is to find a map $F : \overline{\Omega} \times [0,T] \to \mathbb{R}^{n+1}$ such that

$$(\text{MCF})_D \begin{cases} \dfrac{\partial F}{\partial t} = f\nu & \text{in } \Omega \times (0,T) \\ F = F_0 & \text{on } \partial\Omega \times (0,T) \\ F = F_0 & \text{on } \Omega \times \{0\}. \end{cases}$$

where ν is the upward pointing unit normal of $F(\Omega, t)$. An easy ansatz would be to define

$$\tilde{F} : \overline{\Omega} \times [0,T] \to \mathbb{R}^{n+1} : x \mapsto \tilde{F}(x,t) := (x, u(x,t)).$$

However, we can not expect points on the hypersurface only to move in e_{n+1} direction. Therefore, we modify our ansatz by introducing a family of diffeomorphisms $\Phi : \overline{\Omega} \times [0,T] \to \overline{\Omega}$ and defining

$$F : \overline{\Omega} \times [0,T] \to \mathbb{R}^{n+1} : (x,t) \mapsto F(x,t) := \tilde{F}(\Phi(x,t), t).$$

The geometric problem then reads

$$\frac{f}{\sqrt{1 + |Du|^2}} \begin{pmatrix} -Du \\ 1 \end{pmatrix} = f\nu \overset{!}{=} \frac{\partial F}{\partial t} = \frac{\partial \tilde{F}}{\partial t} + (D_x \tilde{F}) \frac{\partial \Phi}{\partial t} = \begin{pmatrix} \phi_t \\ u_t + \langle Du, \phi_t \rangle \end{pmatrix}$$

This yields a parabolic Dirichlet problem for a scalar function u

$$(\text{PDE})_D \begin{cases} \dfrac{\partial u}{\partial t} = \sqrt{1 + |Du|^2} f & \text{in } \Omega \times (0,T) \\ u = u_0 & \text{on } \partial\Omega \times (0,T) \\ u = u_0 & \text{on } \Omega \times \{0\} \end{cases}$$

[1] Note that time dependent Dirichlet boundary conditions do not make sense for an embedding F since it can not be guaranteed that boundary points remain on the boundary under the flow.

and a family of ODEs for Φ which can be solved once u is known

$$\text{(ODE)} \begin{cases} \dfrac{\partial \Phi}{\partial t} = \dfrac{-f}{\sqrt{1+|Du|^2}} Du & \text{in } \Omega \times (0,T) \\ \Phi = \mathrm{id} & \text{on } \Omega \times \{0\}. \end{cases}$$

Remark 8.9 (Compatibility conditions). Note that the first order compatibility conditions are not satisfied in general since the equations force the speed at the boundary of the initial hypersurface to be zero instead of being $\frac{\partial F}{\partial t}\big|_{(\cdot,0)} = f(\cdot,0)\nu_0$ on $\partial\Omega$.

In the case of mean curvature flow we have $\mathbf{H} = -H\nu$ so $f = \operatorname{div}\left(\dfrac{Du}{\sqrt{1+|Du|^2}}\right)$. We obtain the following result.

Proposition 8.10 (Short-time existence: A special case). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{2,\alpha}$-boundary. Let $u_0 \in C^{2,\alpha}(\overline{\Omega})$ and the initial hypersurface M_0 be given as the graph of u_0. Suppose that the mean curvature of M_0 vanishes at the boundary. Then there exists some $T > 0$ such that* (MCF)$_D$ *has a unique solution $F \in C^{2,\alpha;1,\frac{\alpha}{2}}(\overline{Q_T}, \mathbb{R}^{n+1})$.*

Proof. Due to the reasoning above we only need to show that there exists a unique solution $u \in C^{2,\alpha;1,\frac{\alpha}{2}}(\overline{Q_T})$ of (PDE)$_D$. Together with the existence and uniqueness result for ODEs this will yield the desired existence, uniqueness and regularity for the map F. Using the notation of the previous section we are in the situation

$$Q[u] = \sqrt{1+|Du|^2}\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \left(\delta^{ij} - \frac{D^i u D^j u}{1+|Du|^2}\right) D_{ij}u =: a^{ij}(Du)D_{ij}u$$

$$\phi(\cdot,t) := u_0\big|_{\partial\Omega}.$$

Thanks to the additional assumption of vanishing mean curvature on the boundary of the initial hypersurface the compatibility conditions of first order are satisfied, i.e.

$$\phi(\cdot,0) = u_0, \qquad \frac{\partial\phi}{\partial t}\Big|_{t=0} = 0 = -\sqrt{1+|Du_0|^2}H_0 = Q[u_0] \qquad \text{on } \partial\Omega.$$

Furthermore, for the the choice $v := u_0$ we see that $L(Q,v)$ is uniformly parabolic, since

$$L(Q,u_0)w = a^{ij}(Du_0)D_{ij}w + \frac{\partial a^{ij}}{\partial p^k}\Big|_{p=Du_0} D_{ij}u_0 D_k w$$

with

$$\frac{1}{1+|Du_0|^2} \le a^{ij}(Du_0)\xi_i\xi_j \le 2 \qquad \forall\, \xi \in \mathbb{S}^n.$$

Since $v = \phi$ on $\partial\Omega \times (0,T)$ and $v = u_0$ on $\Omega \times \{0\}$ hold by definition the result follows from Theorem 8.2. Uniqueness for solutions of (PDE)$_D$ follows from the comparison principle for linear parabolic PDEs since given two solutions $u, v \in C^{2,1}(\overline{Q}_T)$ we have $w := u - v = 0$ on the parabolic boundary and

$$\begin{aligned} \frac{\partial w}{\partial t} - a^{ij}[Du]D_{ij}w &= \left(a^{ij}[Du] - a^{ij}[Dv]\right)D_{ij}v \\ &= \left(\int_0^1 \frac{\partial a^{ij}}{\partial p^k}\Big|_{sDu+(1-s)Dv} ds\, D_{ij}v\right) D_k w =: b^k D_k w. \end{aligned}$$

Thus w is a solution of a linear parabolic problem with $L := \frac{\partial}{\partial t} - a^{ij}[v(x)]D_{ij} + b^k(x)$. □

Remark 8.11. As mentioned in the previous section one can use weighted Hölder spaces to prove existence of the nonlinear Dirichlet problem without requiring the first order compatibility condition, i.e. without demanding $H_0 = 0$ on the boundary. In fact one can show much more, namely that the flow exists for all times and the hypersurfaces converge to the minimal surface spanned by the boundary values.

The evolution of graphs under mean curvature flow with Dirichlet and Neumann boundary conditions was investigated by Huisken in [33]. Let us state the results without referring to the weighted spaces.

Theorem 8.12 (Convergence to a minimal surface). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{2,\alpha}$-boundary of positive mean curvature. Let $u_0 \in C^{2,\alpha}(\overline{\Omega})$. Then the problem* $(\mathrm{PDE})_D$ *has a unique solution for all time which is smooth in $\Omega \times (0,T)$ and satisfies*

$$u(\cdot,t) \in C^{2,\alpha}(\overline{\Omega}) \quad \forall\, t > 0, \qquad u(x,\cdot) \in C^{1,\frac{\alpha}{2}}([0,\infty)) \quad \forall\, x \in \Omega' \subset\subset \Omega.$$

Furthermore

$$\lim_{t\to\infty} \|u(\cdot,t) - \overline{u}\|_{C^{2,\beta}(\overline{\Omega})} = 0, \qquad \beta < \alpha$$

where $\overline{u}$ is the minimal hypersurface with Dirichlet boundary values $u_0\big|_{\partial\Omega}$.

Proof. See [33]. As in the elliptic case the long-time existence will follow from a priori estimate together with a parabolic version of the Schauder theory. A main ingredient is the gradient estimate. Since the Dirichlet boundary values are independent of t one can use similar barriers as in the elliptic case. Unfortunately, we don't have time to discuss the a priori estimates which yield the long-time existence and convergence. In order to see that the limiting hypersurface has to have zero mean curvature one computes that

$$\frac{d}{dt} area\big(F(\Omega,\cdot)\big) = -\int_{F(M,\cdot)} \mathrm{div}_{F(M,\cdot)}(-H\nu)\, d\mu = -\int_\Omega H^2 \sqrt{1+|Du|^2}\, dx.$$

Integrating this equality from $t = 0$ to infinity and taking the gradient estimate into account shows that H has to converge to zero as t tends to infinity. □

A similar result holds for the corresponding Neumann problem with ninety degree contact angle, i.e. for the problem

$$(\mathrm{MCF})_N \begin{cases} \dfrac{\partial F}{\partial t} = \mathbf{H} & \text{in } \Omega \times (0,T) \\ \langle \nu, \mu \rangle = 0 & \text{on } \partial\Omega \times (0,T) \\ F = F_0 & \text{on } \Omega \times \{0\}. \end{cases}$$

The corresponding scalar problem is

$$(\mathrm{PDE})_N \begin{cases} \dfrac{\partial u}{\partial t} = \sqrt{1+|Du|^2}\, \mathrm{div}\left(\dfrac{Du}{\sqrt{1+|Du|^2}}\right) & \text{in } \Omega \times (0,T) \\ D_\mu u = 0 & \text{on } \partial\Omega \times (0,T) \\ u = u_0 & \text{on } \Omega \times \{0\} \end{cases}$$

Remark 8.13 (Invariance of $\partial\Omega$). The compatibility condition of zero order is satisfied if initially $\langle \nu_0, \mu\rangle = 0$. Furthermore, the diffeomorphisms $\Phi(\cdot,t)$ keep $\partial\Omega$ invariant since

$$\left\langle \frac{\partial \Phi}{\partial t}, \mu \right\rangle = \left\langle \frac{\partial F}{\partial t}, \begin{pmatrix} \mu \\ 0 \end{pmatrix} \right\rangle = \left\langle f\nu, \begin{pmatrix} \mu \\ 0 \end{pmatrix} \right\rangle = 0.$$

Note that the same was true for the Dirichlet problem with time independent boundary data since there $\frac{\partial F}{\partial t} = 0$ on $\partial\Omega$.

The following result holds.

Theorem 8.14 (Convergence to a piece of a plane). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{2,\alpha}$-boundary of positive mean curvature. Let $u_0 \in C^{2,\alpha}(\overline{\Omega})$ such that $D_\mu u_0 = 0$ on $\partial\Omega$. Then the problem* $(\mathrm{PDE})_N$ *has a unique solution $u \in C^{2,\alpha;1,\frac{\alpha}{2}}(\overline{Q_\infty})$. Furthermore*

$$\lim_{t\to\infty} \|u(\cdot,t)\|_{C^{2,\beta}(\overline{\Omega})} = 0, \qquad \beta < \alpha.$$

Proof. The proof is also contained in [33]. □

Exercise III.1. Show that $(\mathrm{PDE})_N$ has a solution at least for a short time.

Remark 8.15 (Entire graphs). Ecker-Huisken [13] proved a long-time existence result for MCF of surfaces defined as the graph of a function $u : \mathbb{R}^n \to \mathbb{R}$. Those hypersurfaces are called entire graphs.

One can also consider graphs over $\mathbb{S}^n$ instead of graphs over domains in $\mathbb{R}^n$. A classical result in that setting is the long-time existence and convergence of convex hypersurfaces under mean curvature flow, proved by Huisken in 1984 in the case $n \geq 2$ and by Gage-Hamilton in 1986 for $n = 1$.

Theorem 8.16 (Flow by mean curvature of convex surfaces into spheres.). *Let $n \geq 2$ and assume that M_0 is uniformly convex, i.e. the eigenvalues of its second fundamental form are strictly positive everywhere. Then the area preserving mean curvature flow, i.e.*

$$\tilde{F}(\cdot,t) := \psi(t)F(\cdot,t) \quad s.t. \quad |\tilde{F}(M,t)| = |M_0|, \quad \frac{\partial F}{\partial t} = \mathbf{H}, \quad F(M,0) = M_0$$

has a solution for all time and the surfaces converge to a sphere of area $|M_0|$ smoothy.

Proof. See [18] for $n = 1$ and [31] for $n \geq 2$. □

Remark 8.17 (Explicit solutions of MCF).

i) Spheres: Due to the symmetry, the evolution of round spheres under MCF can be computed explicitly. Starting with a sphere of radius r_0 in $\mathbb{R}^{n+1}$ the radius at time t is given by $r(t) = \sqrt{r_0^2 - 2nt}$. So the flow will exist until $T = r_0^2/2n$ where the spheres shrink to a point.

ii) Cylinders: Similarly, as for spheres, a cylinder $\mathbb{S}^{n-k}_{r_0} \times \mathbb{R}^k$ remains round and has at time t the radius $r(t) = \sqrt{r_0^2 - 2(n-k)t}$. So it shrinks to a line in time $r_0^2/2(n-k)$.

iii) Tori: In general the speed is different in different directions. Therefore, they don't keep their shape and a solution can't be computed as the solution of an ODE. However, one can show that a torus $\mathbb{S}^{n-k} \times \mathbb{S}^k$ of positive mean curvature will shrink to a circle in finite time. Angenent even found a torus which is a self-similar solution, i.e. which keeps it shape during the flow.

Another classical (1987) result by Grayson concerns MCF of curves in $\mathbb{R}^2$ also called curve-shortening flow.

Theorem 8.18 (Curve-shortening flow). *Every smooth, embedded curve in $\mathbb{R}^2$ which evolves under the curve-shortening flow becomes convex in finite time.*

Proof. See Grayson [25]. □

Furthermore, there is a comparison principle for MCF:

Proposition 8.19 (Comparison principle). *If M_0 and $\tilde{M}_0$ are disjoint the the same holds for M_t and $\tilde{M}_t$, i.e. for the evolving hypersurfaces under MCF.*

Proof. R. Hamilton found a nice geometric way to prove this result. He showed that $\frac{d}{dt}\,\mathrm{dist}(M_t, \tilde{M}_t) \geq 0$. □

This implies the following interesting result.

Corollary 8.20 (All embedded planar curves shrink to points). *Every smooth, embedded curve in $\mathbb{R}^2$ which evolves under the curve-shortening flow shrinks to a point in less than $T_{max} = R^2/2$ where R is the radius of a circle which encloses the initial curve.*

Proof. By Graysons theorem any curve becomes convex in finite time. Then by Gage-Hamilton the curve shrinks to a point. Due to the comparison principle we can argue: if the trace of the initial curve is contained in a circle it must shrink to a point faster that the circle which needs the time $R^2/2$. □

Remark 8.21 (Singularities of MCF). In general the evolving surfaces might develop singulars under the flow. Image two large spheres which are connected by a thin cylindrical tube. Then the tube will shrink of faster that the spheres which causes a so called neck-pinch or a cusp depending on the ratio of the two spheres.

In the case of 2-convex hypersurfaces[2] the nature of the singularities is understood. Huisken-Sinestrari [35] showed that in this case every singularity can be rescaled to a shrinking sphere, a shrinking cylinder or a translating solution. This leads tothe following result

Theorem 8.22 (Classification of 2-convex hypersurfaces). *Let $n \geq 2$. If $M^n \subset \mathbb{R}^{n+1}$ is a smooth, compact, 2-convex hypersurface. Then M^n is diffeomorphic either to $\mathbb{S}^n$ or to a finite connected sum of $\mathbb{S}^{n-1} \times \mathbb{S}$.*

[2]A hypersurface is called k-convex if the sum of the smallest k Eigenvalues of the second fundamental form are positive.

Proof. See [35] for $n \geq 3$. The proof is very involved. The main idea is to first understand the singularities which can possibly occur under MCF of 2-convex hypersurfaces. Once they are classified one starts the flow with $M_0^n := M^n$. The the flow is stopped before the singular time and the region of high curvature is replaced by a spherical cap. Then one restarts the flow of the individual pieces and continues this procedure (finitely many times) until only pieces of $\mathbb{S}^n$ and and $\mathbb{S}^{n-1} \times \mathbb{S}$ are left. The procedure is called surgery and is inspired by the surgery procedure for Ricci flow (see Hamilton [27,28]). It is somehow the inverse operation of tatting the connected sum (which is done to reconstruct the initial surface from the final pieces).

The case $n = 2$ was only settled 2013 by Huisken-Sinestrari based on a non-collapsing result by Andrews [2] and work by Haslhofer-Kleiner [29]. □

Remark 8.23 (**Classification of abstract three manifolds**)**.** If M^3 is an abstract three manifold of positive Ricci curvature (implies 2-convexity) which can be isometrically embedded into $\mathbb{R}^4$ then the above classification result applies. By proving the geometrization conjecture of Thurston [62], Perelman [50–52] obtained a classification of all three manifolds (without the assumption that they can be isometrically embedded into $\mathbb{R}^4$).

In general there are only partial results about the singularities of hypersurfaces under MCF. There are three succesful approaches to the problem. One is using classical differential geometry and PDE theory as in Huisken-Sinestrari [35] a second is using geometric measure theory (see for example White [65, 66] and Ilmanen [36]) and the third is using the level set approach (see Evans-Spruck [15, 16] and Chen-Giga-Goto [8]) which we will briefly discuss in the next chapter.

8.3. A Neumann problem for inverse mean curvature flow

In this section we want to focus on the evolution of hypersurfaces under inverse mean curvature flow, i.e.

$$F : M^n \times [0, T) \to (N^{n+1}, \overline{g}) \quad \text{s.t.} \quad \frac{\partial F}{\partial t} = \frac{1}{H}\nu.$$

The easiest examples are again round spheres. Starting with a sphere of radius r_0 in $\mathbb{R}^{n+1}$ yields a sphere of radius $r(t) = r_0 e^{t/n}$ at time t. In this case the hypersurfaces are expanding. Note that in the special case of round spheres the flow exists for all time. That this is not true in general can be seen by starting from a thin torus of strictly positive mean curvature.

In the following we want to consider hypersurfaces with boundary which move along but stay perpendicular to a fixed supporting hypersurface $\Sigma^n \subset (N^{n+1}, \overline{g})$. So we want to consider the problem

$$(\text{IMCF})_N \begin{cases} \dfrac{\partial F}{\partial t} = \dfrac{1}{H}\nu\Big|_F & \text{in } M^n \times (0, T) \\ \overline{g}(\nu, \mu)\big|_F = 0 & \text{on } \partial M^n \times (0, T) \\ F = F_0 & \text{on } M^n \times \{0\} \end{cases}$$

where μ is the unit normal of Σ^n and ν is the unit normal of $F(M^n,t)$. As a compatibility condition we require that

$$F_0(\partial M^n) = F_0(M^n) \cap \Sigma^n, \qquad \langle \nu_0, \mu \circ F_0 \rangle = 0 \text{ on } \partial M^n$$

holds for the initial immersion. Note that for ambient spaces different than $\mathbb{R}^{n+1}$ the expression $\partial F/partialt$ has to be interpreted as the push forward of $\partial/partialt$ by $F_\star$.

Remark 8.24. The corresponding Neumann problem for MCF was first studied by Stahl [57–59]. Note that for $(\mathrm{MCF})_N$ and $(\mathrm{IMCF})_N$ short-time existence holds for immersed initial hypersurfaces of strictly positive mean curvature in Riemannian ambient manifolds (see [43, 57]). The same results hold for closed hypersurfaces.

Here we want to focus on a setting in which we will be able to prove long-time existence and convergence. The corresponding problem for closed star-shaped hypersurfaces under IMCF was first studied by Gerhardt [21] and Urbas [63].

Definition 8.25 (A special choice for Σ^n). In the following we assume that the supporting hypersurface is a smooth cone, i.e.

$$\Sigma^n := \{\ rp \in \mathbb{R}^{n+1} \mid r > 0, \quad p \in \partial M^n\ \}, \qquad M^n \subset \mathbb{S}^n \subset \mathbb{R}^{n+1} \text{ smooth}$$

with outward pointing unit normal μ. We will say that Σ^n is a cone over M^n. Furthermore, we suppose that the initial hypersurface M_0^n is star-shaped with respect to the center of the cone, and touches it in a ninety degree angle. Note that the star-shapedness implies that we can represent M_0^n as a graph of a function over $M^n \subset \mathbb{S}^n$.

Before we state the sort-time existence result let us compute the mean curvature of graphs over $M^n \subset \mathbb{S}^n$.

Definition 8.26 (Graphs over $\mathbb{S}^n$ I). We consider embeddings F which are given as graphs over the sphere, i.e.

$$F : M^n \subset \mathbb{S}^n \subset \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} : p \mapsto F(p) := u(p)p, \qquad u : \mathbb{S}^n \to \mathbb{R} : p \mapsto u(p).$$

In the following we denote by σ_{ij} the coefficients of the metric σ on $\mathbb{S}^n$ with respect to a coordinate basis ∂_i and set $\tau_i := F_\star \circ \partial_i \circ F^{-1}$. Furthermore, we put

$$\nabla_i u := \partial_i u, \qquad |\nabla u|^2 := \sigma^{ij} \nabla_i u \nabla_j u, \qquad \nabla_{ij} u := (\mathrm{Hess}_{\mathbb{S}^n} u)_{ij}\,.$$

Finally, the exterior unit normal of $\mathbb{S}^n$ is denoted by n and the upper unit normal of $F(M^n)$ is denoted by ν.

Lemma 8.27 (Graphs over $\mathbb{S}^n$ II). *Using the definitions above we obtain the following formulae*

i) Let $v := \sqrt{1 + u^{-2}|\nabla u|^2}$. For $1 \le i \le n$ we have

$$\tau_i \circ F = n \nabla_i u + u \partial_i, \qquad \nu \circ F = \frac{1}{v}\left(n - \frac{\nabla^i u}{u}\partial_i\right).$$

ii) *With respect to the basis τ_i the metric and inverse metric on $TF(M^n)$ have the coefficients*

$$g_{ij}\circ F = u^2\sigma_{ij} + \nabla_i u\nabla_j u, \qquad g^{ij}\circ F = \frac{1}{u^2}\left(\sigma^{ij} - \frac{\nabla^i u\nabla^j u}{u^2+|\nabla u|^2}\right).$$

iii) *With respect to the same basis the second fundamental form has the coefficients*

$$h_{ij}\circ F = \frac{u}{v}\Big(\sigma_{ij} + 2u^{-2}\nabla_i u\nabla_j u - u^{-1}\nabla_{ij}u\Big).$$

Note that ∇ denotes covariant derivatives on $(\mathbb{S}^n,\sigma)$.

iv) *The condition on the contact angle translates as follows*

$$\langle\check{\mu},\nu\rangle_{\mathbb{R}^{n+1}}\big|_{F(p)} = 0 \;\Leftrightarrow\; \nabla_\mu u := \sigma_{ij}\mu^i\nabla^j u = 0$$

where $\check{\mu}$ is the normal of Σ^n at $F(p)$ and μ is the normal of Σ^n at p i.e. $\mu = \mu^i\partial_i$.

Proof. We will only sketch the proof. Please make sure that you can fill in the gaps.

i) Choose a chart $\phi : U \subset \mathbb{R}^n \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$ and use the corresponding map

$$F_\phi : U \to \mathbb{R}^{n+1} : \theta \mapsto F_\phi(\theta) := u\big(\phi(\theta)\big)\phi(\theta)$$

together with $\partial_i = \phi_\star \circ e_{\theta_i} \circ \phi^{-1}$ to compute $\tau_i \circ F = (F_\phi)_\star \circ e_{\theta_i}$. The formula for ν can be checked by verifying $\langle\nu,\nu\rangle = 1$ and $\langle\nu,\tau_i\rangle = 0$.

ii) The formula for the coefficients of the metric follows directly from $g_{ij} = \langle\tau_i,\tau_j\rangle_{\mathbb{R}^{n+1}}$. The formula for the coefficients of the inverse metric can be checked by verifying that $g^{ij}g_{jk} = \delta^i_k$.

iii) The formula for the coefficients of h follows from the formula which we derived in Chapter 2, Section 2.2 about isometric immersions. Note that in $\mathbb{R}^{n+1}$ the ambient Christoffel symbols vanish so we obtain

$$h_{ij}\circ F = -\left\langle {}^F D_{\partial_i}\left(F_\star\circ\partial_j\right), \nu\circ F\right\rangle_{\mathbb{R}^{n+1}}.$$

Using the formulae for τ_j and ν and the fact that $\mathbb{S}^n$ is umbilic, i.e. $h(\mathbb{S}) = \sigma$ the result follows by a direct computation.

iv) This follows since Σ^n has a straight direction and therefore the outward unit normal at p and $F(p)$ point in the same direction.

□

Exercise III.2 (Graphs over $\mathbb{S}^n$). Try to fill in the gaps in the proof of Lemma 8.19.

Remark 8.28 (Transformation to $w := \log u$). It will be convenient to express all quantities in terms of $w := \log u$. We obtain $v = \sqrt{1+|\nabla w|^2}$ and compute that

$$g_{ij}\circ F = e^{2w}\sigma_{ij} + \nabla_i w\nabla_j w, \qquad h_{ij}\circ F = \frac{1}{ve^w}\left(\sigma_{ij} + \nabla_i w\nabla_j w - \nabla_{ij}w\right),$$

$$g^{ij}\circ F = e^{-2w}\left(\sigma^{ij} - \frac{\nabla^i w\nabla^j w}{1+|\nabla w|^2}\right), \qquad H\circ F = \frac{1}{ve^w}\left[n - \left(\sigma^{ij} + \frac{\nabla^i w\nabla^j w}{1+|\nabla w|^2}\right)\nabla_{ij}w\right].$$

As in the case of graphs in $\mathbb{R}^n$ let us reduce $(\text{IMCF})_N$ to a scalar parabolic problem for the height function $u : M^n \subset \mathbb{S}^n \to \mathbb{R}$. We make the ansatz

$$\tilde{F} : M^n \times [0,T) \to \mathbb{R}^{n+1} : (p,t) \mapsto u(p,t)p,$$

$$\Phi : M^n \times [0,T) \to M^n : (p,t) \mapsto \Phi(p,t),$$

$$F : M^n \times [0,T) \to \mathbb{R}^{n+1} : (p,t) \mapsto \tilde{F}(\Phi(p,t),t).$$

This leads to

$$\frac{1}{Hv}\Big(n - \nabla^i w \partial_i\Big) = \frac{1}{H}\nu\Big|_F \overset{!}{=} \frac{dF}{dt} = \frac{\partial \tilde{F}}{\partial t} + \nabla \tilde{F}\frac{\partial \Phi}{\partial t} = \frac{\partial u}{\partial t} n + \Big((\nabla_k u)n + u\partial_k\Big)\frac{\partial \Phi^k}{\partial t}.$$

Comparing the normal and tangential parts we finally obtain

$$(\text{ODE})\begin{cases} \dfrac{d\Phi}{dt} = \dfrac{-1}{(uvH)\circ F}\nabla w & \text{in } M^n \times (0,T) \\ \Phi = \text{id} & \text{on } M^n \times \{0\} \end{cases}$$

and in terms of $w = \log u$

$$(\text{PDE})\begin{cases} \dfrac{\partial w}{\partial t} = \dfrac{1 + |\nabla w|^2}{n - \left(\sigma^{ij} + \dfrac{\nabla^i w \nabla^j w}{1+|\nabla w|^2}\right)\nabla_{ij} w} & \text{in } M^n \times (0,T) \\ \nabla_\mu w = 0 & \text{on } \partial M^n \times (0,T) \\ w = \log u_0 & \text{on } M^n \times \{0\}. \end{cases}$$

The advantage of writing the PDE in terms of w is that the right hand side only depends on ∇w and $\nabla^2 w$ but not on w. Now we can state the short-time existence result.

Proposition 8.29 (**Short-time existence for** $(\text{IMCF})_N$). *Let $n \geq 2$. Let $M^n \subset \mathbb{S}^n \subset N^{n+1} := \mathbb{R}^{n+1}$ be a domain with smooth boundary and Σ^n a cone over M^n. Let $F_0(M^n) := M_0^n$ be a $C^{2,\alpha}$-hypersurface of strictly positive mean curvature which is star-shaped with respect to the center of the cone. If M_0^n touches Σ^n orthogonally then there exists some $T > 0$ and a unique solution*

$$F \in C^{2,\alpha;1,\frac{\alpha}{2}}(\overline{M^n} \times [0,T]) \cap C^\infty(\overline{M^n} \times (0,T))$$

of $(\text{IMCF})_N$.

Proof. Exercise. □

Exercise III.3 (**Short-time existence of** $(IMCF)_N$ **in a cone**). Show that (PDE) has a unique solution at least for a short time. What else is needed to actually prove short-time existence of $(IMCF)_N$?

Remark 8.30 (**Short-time existence in a general setting**). Note that short-time existence also holds for arbitrary smooth supporting hypersurfaces Σ^n in a Riemannian ambient manifold and immersed initial hypersurfaces M_0^n of strictly positive mean curvature which touch Σ^n orthogonally[3]. Let us assume that M_0^n is embedded. In that situation one proceeds as follows:

[3] If you are willing to work with weighted Hölder spaces you can even drop this compatibility condition of having a ninety degree contact angle initially.

a) Construct a vector field in a neighborhood of $M_0^n \subset (N, \overline{g})$ which is tangential to Σ^n along Σ^n and normal to M_0^n along M_0^n.

b) Argue that this gives rise to a generalized tubular neighborhood[4] $\mathcal{U}$ around M_0^n via the flow lines of this vector field.

c) Consider an isometric immersion Φ between $M^n \times (-\varepsilon, \varepsilon)$ and $\mathcal{U}$.

d) For small $t > 0$ this allows us to describe $F(M^n, t) \subset \mathcal{U}$ as a graph in $\big(M^n \times (-\varepsilon, \varepsilon), \Phi^\star \overline{g}\big)$, i.e. $\operatorname{graph} u = \{(x, u(x,t)) \subset M^n \times (-\varepsilon, \varepsilon) \mid x \in M^n\}$.

e) Finally, one derives formulae similar to those of Lemma 8.27 and splits the problem into an ODE and a scalar parabolic PDE for u.

Think about how to modify this procedure to be applicable for immersed initial hypersurfaces M_0^n?

Definition 8.31. Let us set

$$Q : \mathbb{R}^n \times \mathbb{R}^{n\times n} : (p,q) \mapsto Q(p,q) := \frac{1+|p|^2}{n - \left(\sigma^{ij} - \dfrac{p^i p^j}{1+|p|^2}\right) q_{ij}}.$$

Note that Q is a nonlinear second order operator but in contrast to the equation for u there is no dependence on the function itself. We will use the following notation

$$Q_{q^{ij}}(p,q) := \left.\frac{\partial Q}{\partial q^{ij}}\right|_{(p,q)}, \qquad Q_{p^k}(p,q) := \left.\frac{\partial Q}{\partial p^k}\right|_{(p,q)}$$

and see that

$$\begin{aligned} Q_{q^{ij}}\big|_{[w]} &:= Q_{q^{ij}}(\nabla w, \nabla^2 w) \\ &= \frac{v^2}{\left[n - \left(\sigma^{ij} - \dfrac{\nabla^i w \nabla^j w}{1+|\nabla w|^2}\right)\nabla^2_{ij} w\right]^2}\left(\sigma^{ij} - \frac{\nabla^i w \nabla^j w}{1+|\nabla w|^2}\right) = \frac{1}{H^2} g^{ij} \end{aligned}$$

is strictly positive definite once we have estimates for H.

In the following we will use the PDE for w to derive estimates for $|u|, |\partial u/\partial t|, |\nabla u|$ and $|H|$. These a priori estimates are obtained for smooth solutions, i.e.:

Definition 8.32 (Admissible solutions). Let $T_{\max} > 0$ be the maximal existence time such that (PDE) has a unique solution $w \in C^{2,\alpha;1,\frac{\alpha}{2}}(M^n \times [0, T_{\max})) \cap C^\infty(M^n \times (0, T_{\max}))$. Those solutions are called admissible solutions.

The idea is to show that admissible solutions (which exist by the short-time existence result) can't blow up in finite time. We start with an estimate for $|u|$.

[4] In the case of closed hypersurfaces or Σ^n being totally geodesic there is no supporting hypersurface Σ^n and one can simply use the classical tubular neighborhood.

Lemma 8.33 (Sup-estimate). *Let w be an admissible solution of* (PDE)*. Let Σ^n be a smooth cone. Then u satisfies*

$$R_1 := \min_{M^n} u_0 \;\leq\; u(x,t)e^{-t/n} \;\leq\; \max_{M^n} u_0 =: R_2$$

for all $(x,t) \in M^n \times [0,T]$.

Proof. Let $w(x,t) := \ln u(x,t)$ and $w^+(x,t) := \ln(\max_{M^n} u_0) + t/n$. Both satisfy the same PDE. Using

$$R^{ij} := \int_0^1 Q_{q^{ij}}(\nabla w_s, \nabla^2 w_s)ds, \qquad S^k := \int_0^1 Q_{p^k}(\nabla w_s, \nabla^2 w_s)ds$$

with $w_s := sw^+ + (1-s)w$, we see that $\psi := w^+ - w$ satisfies

$$\begin{cases} \dfrac{\partial \psi}{\partial t} &= R^{ij}\nabla_{ij}\psi + S^k\nabla_k\psi \qquad \text{in} \quad M^n \times (0,T) \\ \nabla_\mu \psi &= 0 \qquad \text{on} \quad \partial M^n \times (0,T) \\ \psi(\cdot,0) &\geq 0 \qquad \text{on} \quad M^n. \end{cases}$$

The linear parabolic maximum principle implies $\psi \geq 0$ in $M^n \times [0,T]$ and thus the upper bound. The lower bound is obtained in the same way using $w^-(x,t) := \ln(\min_{M^n} u_0) + t/n$. □

Remark 8.34. From a geometric point of view this estimate says that the rescaled surfaces $F(M^n,t)e^{-t/n}$ always stay between the two spherical caps which enclose the initial surface.

Next we want to estimate $\dot{u} := \partial u/\partial t$.

Lemma 8.35 (Time derivative estimate: PDE version). *Let w be an admissible solution of* (PDE)*. Let Σ^n be a smooth cone. Then $\dot{u} := \partial u/\partial t$ satisfies*

$$\left(\frac{R_1}{R_2}\right) \min_{M^n} \frac{v_0}{H_0} \leq \dot{u}(x,t)e^{-t/n} \leq \left(\frac{R_2}{R_1}\right) \max_{M^n} \frac{v_0}{H_0}$$

for all $(x,t) \in M^n \times [0,T]$, where $H_0 = H(\,.\,,0)$, $v_0 = v(\,.\,,0)$ and R_1, R_2 are defined as in Lemma 8.33.

Proof. Let u satisfy (PDE) and $w := \ln u$. Then $\dot{w} := \partial w/\partial t$ satisfies

$$\begin{cases} \dfrac{\partial \dot{w}}{\partial t} &= Q_{q^{ij}}\big|_{[w]}\nabla_{ij}\dot{w} + Q_{p^k}\big|_{[w]}\nabla_k\dot{w} \qquad \text{in} \quad M^n \times (0,T) \\ \nabla_\mu \dot{w} &= 0 \qquad \text{on} \quad \partial M^n \times (0,T) \\ \dot{w}(\cdot,0) &= Q(\nabla w_0, \nabla^2 w_0) \qquad \text{on} \quad M^n \end{cases}$$

with $Q(\nabla w_0, \nabla^2 w_0) \geq 0$. The evolution equation follows directly by differentiating the evolution equation for w with respect to t. The initial value $\dot{w}(\cdot,0)$ is also obtained from the evolution equation of w at time zero. For the Neumann condition we note that $\nabla_\mu w$ is differentiable in t for $t > 0$ and equal to zero for all $t > 0$. Thus,

$$0 = \frac{\partial}{\partial t}(\nabla_\mu w) = \nabla_{\dot\mu} w + \nabla_\mu \dot w = \nabla_\mu \dot w$$

since Σ^n is a cone and thus μ does not depend on t. Therefore, the maximum principle for linear parabolic PDEs implies

$$\min_{M^n} \frac{v_0}{u_0 H_0} = \min_{M^n} \dot w(\cdot, 0) \le \dot w(x,t) \le \max_{M^n} \dot w(\cdot, 0) = \max_{M^n} \frac{v_0}{u_0 H_0}.$$

Using the estimate for u and the fact that $\dot w = u^{-1}\dot u$ we obtain the desired result. □

For the estimate of $|\nabla u|$ we have to make use of the convexity of Σ^n.

Lemma 8.36 (**Gradient estimate: PDE version**). *Let w be an admissible solution of* (PDE). *Let Σ^n be a smooth, convex cone. Then*

$$|\nabla u(x,t)| e^{-t/n} \le \left(\frac{R_2}{R_1}\right) \max_{M^n} |\nabla u_0|$$

for all $(x,t) \in M^n \times [0,T]$.

Proof. As in [21] we want to find a boundary value problem for $\psi := |\nabla w|^2/2$. Therefore, we first calculate

$$\nabla_k \psi = \nabla_{mk} w \nabla^m w, \qquad \nabla_{ij}\psi = \nabla_{mij} w \nabla^m w + \nabla_{mi} w \nabla_j^m w.$$

Using the rule for interchanging covariant derivatives on $\mathbb{S}^n$ together with the Gauss equations, i.e. $R_{ijkl} = h_{il}h_{jk} - h_{ik}h_{jl}$ and the fact that $\mathbb{S}^n$ is umbilic, i.e. $h(\mathbb{S}^n) = \sigma$ we obtain

$$\nabla_{mij} w \;=\; \nabla_{imj} w \;=\; \nabla_{ijm} w + R_{im\ j}^{\ \ \ l} \nabla_l w \;=\; \nabla_{ijm} w + \sigma_{ij}\nabla_m w - \sigma_{im}\nabla_j w$$

which implies

$$\nabla_{ij}\psi = \nabla_{ijm} w \nabla^m w + \sigma_{ij}|\nabla w| - \sigma_{im}\nabla_j w \nabla^m w + \nabla_{mi} w \nabla_j^m w.$$

We only wirte $Q_{q^{ij}}$ instead of $Q_{q^{ij}}\big|_{[w]}$ and obtain

$$\begin{aligned}\dot\psi &= \nabla_m \dot w \nabla^m w \\ &= \nabla_m Q(\nabla w, \nabla^2 w)\nabla^m w \\ &= Q_{q^{ij}} \nabla_{ijm} w \nabla^m w + Q_{p^k}\nabla_{km} w \nabla^m w \\ &= Q_{q^{ij}}\nabla_{ij}\psi - Q_{q^{ij}}\sigma_{ij}|\nabla w| + Q_{q^{ij}}\sigma_{im}\nabla_j w \nabla^m w - Q_{q^{ij}}\nabla_{mi} w \nabla_j^m w + Q_{p^k}\nabla_k \psi.\end{aligned}$$

Using the special form of $Q_{q^{ij}}$ we see that

$$\begin{aligned}&- Q_{q^{ij}}\sigma_{ij}|\nabla w|^2 + Q_{q^{ij}}\sigma_{im}\nabla_j w \nabla^m w \\ &\quad = \frac{1}{u^2H^2}\left(\sigma^{ij} - \frac{\nabla^i w \nabla^j w}{1+|\nabla w|^2}\right)\left(\nabla_i w \nabla_j w - \sigma_{ij}|\nabla w|^2\right) = \frac{(1-n)|\nabla w|^2}{u^2H^2}\end{aligned}$$

and

$$Q_{q^{ij}} \nabla_{mi} w \nabla_j^m w$$
$$= \frac{1}{u^2 H^2} \left(\sigma^{ij} - \frac{\nabla^i w \nabla^j w}{1 + |\nabla w|^2} \right) \nabla_{mi} w \nabla_j^m w = \frac{|\nabla^2 w|^2}{u^2 H^2} - \frac{|\nabla \psi|^2}{u^2 v^2 H^2}.$$

Thus the evolution equation for ψ can be written as

$$\frac{\partial \psi}{\partial t} = Q_{q^{ij}} \nabla_{ij} \psi + \left(Q_{p^k} + \frac{\nabla^k \psi}{u^2 v^2 H^2} \right) \nabla_k \psi - \frac{2(n-1)}{u^2 H^2} \psi - \frac{|\nabla^2 w|^2}{u^2 H^2}. \tag{8.1}$$

For the Neumann condition we use the fact that for $t > 0$ the function $\nabla_\mu w$ is differentiable and $\nabla_\mu w \equiv 0$. Since $\nabla_\mu \psi$ is a coordinate invariant expression (a (0,0)-tensor) we use an orthonormal frame for the calculation. Let $e_1, ..., e_{n-1} \in T_x \partial M^n$ and $e_n = \mu$. Then we have

$$\begin{aligned}
\nabla_\mu \psi &= \sum_{i=1}^{n-1} \nabla^2 w(e_i, e_n) \nabla_{e_i} w = \sum_{i=1}^{n-1} \left(e_i(e_n(w)) - (\nabla_{e_i} e_n)(w)\right) \nabla_{e_i} w \\
&= - \sum_{i=1}^{n-1} \left((\nabla_{e_i} e_n)(w)\right)^\top \nabla_{e_i} w = - \sum_{i,j=1}^{n-1} \langle \nabla_{e_i} e_n, e_j \rangle \nabla_{e_i} w \nabla_{e_j} w \\
&= - \sum_{i,j=1}^{n-1} {}^{\partial M^n} h_{ij} \nabla_{e_i} w \nabla_{e_j} w
\end{aligned}$$

with ${}^{\partial M^n} h_{ij}$ being the second fundamental form of the boundary ∂M^n. As initial value we can choose $\psi(\,.\,, 0) = |\nabla w_0|^2/2$. Since Σ^n is convex we see that ψ satisfies the inequalities

$$\begin{cases}
\dfrac{\partial \psi}{\partial t} \le Q_{q^{ij}} \nabla_{ij}^2 \psi + \left(Q_{p^k} + \dfrac{\nabla^k \psi}{u^2 v^2 H^2} \right) \nabla_k \psi & \text{in} \quad M^n \times (0, T) \\
\nabla_\mu \psi \le 0 & \text{on} \quad \partial M^n \times (0, T) \\
\psi(\cdot, 0) = |\nabla w_0|^2/2 & \text{on} \quad M^n.
\end{cases}$$

Using the maximum principle (see Theorem 3.31 and Corollary 3.32) we obtain

$$\psi = \frac{|\nabla w|^2}{2} = \frac{|\nabla u|^2}{2u^2} \le \max_{M^n} \frac{|\nabla w_0|^2}{2} = \max_{M^n} \frac{|\nabla u_0|^2}{2u_0^2}.$$

Together with the estimate for u we obtain the desired result. □

A more geometric way to derive the gradient estimate is to estimate the quantity $f := \langle F, \nu \rangle$. Here we use F to denote the position vector.

Lemma 8.37 (Gradient estimate: Geometric version). *Let F be an admissible solution of* (IMCF). *Let Σ^n be a smooth, convex cone. If the initial hypersurface is star-shaped with respect to the center of the cone, i.e. $0 < R_1 \le \langle F_0, \nu_0 \rangle \le R_2$. Then the hypersurfaces remain star-shaped and satisfy*

$$R_1 \le \langle F, \nu \rangle e^{-t/n} \le R_2$$

for all $(x, t) \in M^n \times [0, T]$.

Proof. We first prove the upper bound using the same argument as Huisken-Ilmanen in [34]. We calculate

$$\frac{\partial |F|^2}{\partial t} = \frac{2}{H}\langle F, \nu\rangle \leq \frac{2|F|}{H} \leq \frac{2|F|^2}{n}.$$

The last inequality follows from the observation that at the point most distant form the origin $H \geq n|F|^{-1}$. From the growth of solutions to this ODE we obtain

$$\langle F, \nu\rangle \leq |F| \leq \max |F(\cdot, 0)| e^{t/n} = \max\langle F_0, \nu_0\rangle e^{t/n} \leq R_2 e^{t/n}.$$

The equality comes from the fact that at the maximum of $|F_0|$ we have $|F_0| = \langle F_0, \nu_0\rangle$.

Lower bound: Exercise. □

Exercise III.4 (Geometric estimate for star-shapedness under (IMCF)$_N$**).** Prove a lower bound for $f := \langle F, \nu\rangle$ under $(IMCF)_N$. Hint: Show that f satisfies the following parabolic problem:

$$\begin{cases} \dfrac{\partial f}{\partial t} &= \dfrac{1}{H^2}\Delta_g f + \dfrac{|A|^2}{H^2} f \qquad \text{in} \quad M_t^n \times (0,T) \\ {}^g\nabla_\mu f &= f\, {}^{\Sigma^n} h_{\nu\nu} \qquad \text{on} \quad \partial M_t^n \times (0,T) \\ f(\cdot, 0) &= f_0 \qquad \text{on} \quad M_t^n \end{cases}$$

and use the maximum principle to conclude.

Also the time-derivative can be estimated in a more geometric way.

Lemma 8.38 (Time derivative estimate: Geometric version). *Let F be an admissible solution of* (IMCF). *Let Σ^n be a smooth, convex cone and R_1, R_2 be defined as in Lemma 8.37. Then H satisfies*

$$\left(\frac{R_1}{R_2}\right) \min_{M^n} H_0 \;\leq\; H(x,t)e^{t/n} \;\leq\; \left(\frac{R_2}{R_1}\right) \max_{M^n} H_0$$

for all $(x,t) \in M^n \times [0,T]$.

Proof. Exercise. □

Exercise III.5 (Geometric estimate for the speed under (IMCF)$_N$**).** Prove an a priori estimate for $\dot{w}$. Hint: Show that $\dot{w}$ satisfies the following parabolic problem:

$$\begin{cases} \dfrac{\partial \dot{w}}{\partial t} &= \operatorname{div}_g\left(\dfrac{{}^g\nabla \dot{w}}{H^2}\right) - 2\dfrac{|{}^g\nabla \dot{w}|^2}{\dot{w}H^2} \qquad \text{in} \quad M_t^n \times (0,T) \\ {}^g\nabla_\mu \dot{w} &= 0 \qquad \text{on} \quad \partial M_t^n \times (0,T) \\ \dot{w}(\cdot, 0) &= \dot{w}(\cdot, 0) \qquad \text{on} \quad M_t^n. \end{cases}$$

and use the maximum principle and the previous estimates to conclude.

Remark 8.39 (Rescaling). Note that the surfaces M_t^n tend to infinity as time tends to infinity. From the estimate for u we see that rescaling by the factor $e^{-t/n}$ implies a bound on u. Therefore, we can only expect good estimates for the rescaled solution $\hat{u} = ue^{-t/n}$ or in terms of $w = \ln u$ for $\hat{w} := w - t/n$.

We want to summarize the scaling of the important quantities in the next Lemma.

Lemma 8.40. *Let F be a solution of* (IMCF)$_N$*. We obtain the rescaled solution by defining $\hat{F} := Fe^{-t/n}$. This implies the following rescalings*

$$\hat{u} = ue^{-t/n}, \qquad \nabla\hat{u} = \nabla ue^{-t/n}, \qquad \frac{\partial\hat{u}}{\partial t} = \left(\frac{\partial u}{\partial t} - \frac{u}{n}\right)e^{-t/n},$$

$$\hat{w} = w - \frac{t}{n}, \qquad \nabla\hat{w} = \nabla w, \qquad \frac{\partial\hat{w}}{\partial t} = \frac{\partial w}{\partial t} - \frac{1}{n},$$

$$\hat{g}_{ij} = g_{ij}e^{-2t/n}, \qquad \hat{g}^{ij} = g^{ij}e^{2t/n}, \qquad \hat{h}_{ij} = h_{ij}e^{-t/n}, \qquad \hat{H} = He^{t/n}.$$

Proof. From the definition of F we see that the rescaling of F implies the rescaling for u. The other formulae follow by a direct calculation. □

Next, we will prove higher order a priori estimates.

We will first prove estimates for the Hölder coefficients of $\nabla\hat{u}$ and $\partial\hat{u}/\partial t$. They imply a Hölder estimate for the mean curvature $\hat{H}$ which will finally yield the full $C^{2,\alpha;1,\frac{\alpha}{2}}$-estimate for $\hat{u}$. We start with the estimate for the gradient.

Lemma 8.41. *Let w be an admissible solution of* (PDE)*. Let Σ^n be a smooth, convex cone. Then there exists some $\beta > 0$ such that the rescaled function $\hat{u}(x,t) := u(x,t)e^{-t/n}$ satisfies*

$$[\nabla\hat{u}]_{x,\beta} + [\nabla\hat{u}]_{t,\frac{\beta}{2}} \;\leq\; C.$$

Here $[f]_{z,\gamma}$ denotes the γ-Hölder coefficient of f in $M^n \times [0,T]$ with respect to the z-variable and $C = C\left(\|u_0\|_{C^{2,\alpha}(M^n)}, n, \beta, M^n\right)$.

Proof. First note that the a priori estimates for $|\nabla u|$ and $|\partial\hat{u}/\partial t|$ imply a bound for $[\hat{u}]_{x,\beta}$ and $[\hat{u}]_{t,\frac{\beta}{2}}$. The bound for $[\nabla\hat{u}]_{t,\frac{\beta}{2}}$ follows from a bound for $[\hat{u}]_{t,\frac{\beta}{2}}$ and [38], Chapter 2, Lemma 3.1 once we have a bound for $[\nabla\hat{u}]_{x,\beta}$. As $\nabla\hat{u} = \hat{u}\nabla w$ it is enough to bound $[\nabla w]_{x,\beta}$. To get this bound we fix t and rewrite (PDE) as an elliptic Neumann problem:

$$\operatorname{div}_\sigma\left(\frac{\nabla w}{\sqrt{1+|\nabla w|^2}}\right) = f := \frac{n}{\sqrt{1+|\nabla w|^2}} - \frac{\sqrt{1+|\nabla w|^2}}{\dot{w}}. \tag{8.2}$$

Since $\dot{w}$ and $|\nabla w|$ are bounded one can prove a Morrey estimate for ∇w which implies an estimate for $[\nabla\hat{u}]_{x,\beta}$. For the details see [43, 44]. □

In the next step we estimate the Hölder coefficients for $\partial\hat{u}/\partial t$.

Lemma 8.42. *Let w be an admissible solution of* (PDE)*. Let Σ^n be a smooth, convex cone. Then there exists some $\beta > 0$ such that the rescaled function $\hat{u}(x,t) := u(x,t)e^{-t/n}$ satisfies*

$$\left[\frac{\partial \hat{u}}{\partial t}\right]_{x,\beta} + \left[\frac{\partial \hat{u}}{\partial t}\right]_{t,\frac{\beta}{2}} \leq C.$$

Here $[f]_{z,\gamma}$ denotes the γ-Hölder coefficient of f in $M^n \times [0,T]$ with respect to the z-variable and $C = C\left(\|u_0\|_{C^{2,\alpha}(M^n)}, n, \beta, M^n\right)$.

Proof. Similar to the last proof we want to find an appropriate PDE and use the weak formulation. This time we exploit the parabolic equation for $\dot{w}$. We want to follow the argument in [38], Chapter 5, §7 pages 478 ff. We first note that $\dot{w} = v/(uH)$ and therefore

$$\frac{\partial \hat{u}}{\partial t} = \left(\frac{\partial e^w}{\partial t} - \frac{u}{n}\right)e^{-t/n} = \frac{\partial w}{\partial t}e^w e^{-t/n} - \frac{\hat{u}}{n} = \hat{u}\left(\dot{w} - \frac{1}{n}\right).$$

So the estimate for $\dot{w}$ will imply the estimate for $\partial\hat{u}/\partial t$. Recall that $\dot{w}$ satisfies the evolution equation

$$\frac{\partial \dot{w}}{\partial t} = \operatorname{div}_{\hat{g}}\left(\frac{\nabla \dot{w}}{\hat{H}^2}\right) - \frac{2|\nabla \dot{w}|^2_{\hat{g}}}{\dot{w}\hat{H}^2}.$$

In a similar way as the boundedness of f implied a Morrey estimate for ∇w (see previous lemma) the boundedness of $\hat{H}$ and the divergence structure allow us to prove a bound for the Hölder coefficients of $\dot{w}$. For the details see [43, 44].

□

These two estimates directly imply an estimate for the mean curvature.

Lemma 8.43. *Let w be an admissible solution to* (PDE)*. Let Σ^n be a smooth, convex cone. Then there exists some $\beta > 0$ such that the rescaled mean curvature $\hat{H} = He^{t/n}$ satisfies*

$$\left[\hat{H}\right]_{x,\beta} + \left[\hat{H}\right]_{t,\frac{\beta}{2}} \leq C.$$

Here $[f]_{z,\gamma}$ denotes the γ-Hölder coefficient of f in $M^n \times [0,T]$ with respect to the z-variable and $C = C\left(\|u_0\|_{C^{2,\alpha}(M^n)}, n, \beta, M^n\right)$.

Proof. This follows from the fact that

$$\hat{H} = He^{t/n} = \frac{\sqrt{1+|\nabla w|^2}}{e^w \dot{w}}e^{t/n} = \frac{\sqrt{1+|\nabla w|^2}}{\hat{u}\dot{w}}$$

together with the Hölder estimates for $|\nabla w|$, $\dot{w}$ and $\hat{u}$. Note that the Hölder estimate for $\hat{u}$ follows trivially from the estimates on $|\nabla \hat{u}|$ and $|\partial\hat{u}/\partial t|$. □

Finally we obtain the full second order a priori estimates.

Lemma 8.44. *Let w be an admissible solution of* (PDE)*. Let Σ^n be a smooth, convex cone. Then there exists some $\beta > 0$ such that*

$$\|u\|_{C^{2,\beta;1,\frac{\beta}{2}}(M^n \times [0,T])} \leq C$$

with $C = C\left(\|u_0\|_{C^{2,\alpha}(M^n)}, n, \beta, M^n\right)$.

Proof. Recall that $v = \sqrt{1 + |\nabla w|^2}$ and use the formula for the mean curvature to write

$$uvH = n - \left(\sigma^{ij} - \frac{\nabla^i w \nabla^j w}{1 + |\nabla w|^2} \right) \nabla^2_{ij} w = n - u^2 \Delta_g w.$$

Thus we obtain

$$\frac{\partial w}{\partial t} = \frac{v}{uH} = -\frac{uv}{u^2 H^2} H + \frac{2v}{uH} = \frac{1}{\hat{H}^2} \Delta_{\hat{g}} w + \left(\frac{2v}{\hat{u}\hat{H}} - \frac{n}{\hat{u}^2 \hat{H}^2} \right)$$

which is a linear, uniformly parabolic equation with Hölder continuous coefficients. Therefore the linear parabolic theory yields the result. □

Higher order estimates now also follow from the linear parabolic theory:

Lemma 8.45. *Let w be an admissible solution to* (PDE). *Let Σ^n be a smooth, convex cone. Then there exists some $\beta > 0$ and some $t_0 > 0$ such that for all $k \in \mathbb{N}$*

$$\|u\|_{C^{2k,\beta;k,\frac{\beta}{2}}(M^n \times [t_0,T])} \leq C$$

where $C = C\left(\|u(\cdot, t_0)\|_{C^{2k,\alpha}(M^n)}, n, \beta, M^n \right)$.

Proof. Using the $C^{2,\beta;1,\frac{\beta}{2}}$-estimate from Lemma 8.44 we can consider the equations for $\dot{w}$ and $\nabla_i w$ as linear uniformly parabolic equations on the time interval $[t_0, T]$. At the initial time t_0 all compatibility conditions are satisfied and the initial function $u(\cdot, t_0)$ is smooth. This implies (in two steps) a $C^{3,\beta;1,\frac{\beta}{2}}$-estimate for $\nabla_i w$ and (in one step) a $C^{2,\beta;1,\frac{\beta}{2}}$-estimate for $\dot{w}$. Together this yields the result for $k = 2$. From [40], chapter 4, Theorem 4.3, Exercise 4.5 and the preceding arguments one can see that the constants are independent of T. Higher regularity is proved by induction over k. □

Theorem 8.46 (Expansion in a cone). *Let $n \geq 2$. Let Σ^n be a smooth, convex cone with outward unit normal μ. Let $F_0 : M^n \to \mathbb{R}^{n+1}$ be such that $M_0^n := F_0(M^n)$ is a compact $C^{2,\alpha}$-hypersurface which is star-shaped with respect to the center of the cone and has strictly positive mean curvature. Furthermore, assume that M_0^n meets Σ^n orthogonally, i.e. $F_0(\partial M^n) \subset \Sigma^n$ and $\langle \mu, \nu_0 \circ F_0 \rangle|_{\partial M^n} = 0$ where ν_0 is the unit normal to M_0^n. Then there exists a unique embedding*

$$F \in C^{2,\alpha;1,\frac{\alpha}{2}}(M^n \times [0,\infty), \mathbb{R}^{n+1}) \cap C^\infty(M^n \times (0,\infty), \mathbb{R}^{n+1})$$

with $F(\partial M^n, t) \subset \Sigma^n$ for $t \geq 0$, satisfying (IMCF). *Furthermore, the rescaled embedding $F(\cdot, t)e^{-t/n}$ converges smoothly to an embedding F_∞, mapping M^n into a piece of a round sphere of radius $r_\infty = (|M_0^n|/|M^n|)^{(1/n)}$.*

Proof. By the sort-time existence result we know that an admissible solution with the desired regularity exists at least for a short time. Furthermore, by Lemma 8.45 we see that the Hölder norm of $u = \hat{u}e^{t/n}$ can not blow up at $T^* < \infty$. Therefore, w can be extended to be a solution to (PDE) in $[0, T^*]$ and the short-time existence result implies the existence of a solution beyond T^* which is a contradiction to the choice of T^*. Thus $T^* = \infty$.

To investigate the rescaled embedding as t tends to infinity we have to examine the behavior of $\hat{u} = ue^{-t/n}$. The a priori estimates allow us to rewrite the PDE for $\psi := |\nabla w|^2/2$ as

$$\frac{\partial \psi}{\partial t} \;\leq\; Q^{ij}\nabla_{ij}\psi + B^k\nabla_k\psi - \gamma\psi.$$

with some $\gamma > 0$ which implies an exponential decay of the gradient, i.e.

$$|\nabla \hat{u}| \leq \left(\frac{R_2}{R_1}\right) \max_{M^n} |\nabla u_0| e^{-\gamma t}.$$

Using the formula for the first variation of area we have

$$\frac{\mathrm{d}}{\mathrm{d}t}|M_t^n| = \int_{M_t^n} H \left\langle \nu, \frac{1}{H}\nu \right\rangle \, d\mu_t + \int_{\partial M_t^n} \left\langle \mu, \frac{1}{H}\nu \right\rangle \, d\sigma_t = |M_t^n|.$$

Thus the surface area grows exponentially and the rescaled hypersurfaces have constant surface area. Using the Arzelà-Ascoli theorem and the decay of the gradient we see that every subsequence must converge to a constant function. The constant surface area implies $|M_0^n| = |\hat{M}_\infty^n| = r_\infty^n |M^n|$ and shows that $\hat{u}(\cdot, t)$ is converging in $C^1(M^n)$ to the constant function $\hat{u}_\infty = r_\infty$.

Now assume that $\hat{u}(\cdot, t)$ converges in $C^k(M^n)$ to r_∞. Since $\hat{u}(\cdot, t)$ is uniformly bounded in $C^{k+1,\beta}(M^n)$ by Arzelà-Ascoli there exists a subsequence which converges to r_∞ in $C^{k+1}(M^n)$. Finally every subsequence must converge and the limit has to be r_∞. Thus $\hat{u}(\cdot, t)$ converges in $C^{k+1}(M^n)$. This finishes the induction and shows that the convergence is smooth. $\square$

9. Outlook: Level set flow and weak solutions of (I)MCF

9.1. Derivation of the level set problem

Let us assume that the deformation of a hypersurface M_0^n in $\mathbb{R}^{n+1}$ is given in terms of the embedding $F : M^n \times [0,T] \to \mathbb{R}^{n+1}$ by

$$\frac{\partial F}{\partial t} = f\nu, \qquad F(\cdot, 0) = F_0$$

for some given embedding F_0 such that $M_0^n = F_0(M^n)$ and a scalar function $f : M^n \times [0,T] \to \mathbb{R}$. In particular we are interested in the choice $f = \lambda H^\alpha$.

Starting from this setting we want to find a description of the hypersurfaces $M_t^n = F(M^n, t)$ as the level sets of a function $u : \Omega \subset \mathbb{R}^{n+1} \to \mathbb{R}$. More precisely,

$$M_t^n = \partial\{x \in \mathbb{R}^{n+1} \mid u(x) < t\}.$$

Base on the parabolic PDE for F we want to find a defining equation for u. Therefore, we first observe that the normal to a level set of u is given by $\nu = Du/|Du|$. Furthermore, we know that the mean curvature can be expressed as the divergence of the normal, i.e.

$$H = \operatorname{div} \nu = \operatorname{div}\left(\frac{Du}{|Du|}\right).$$

Now, consider a curve $\gamma : I \subset \mathbb{R} \to \mathbb{R}^{n+1}$ such that $\gamma(t) \in M_t^n$. Then $u(\gamma(t)) = t$ and differentiating this expression with respect to t yields $\langle Du, \dot\gamma(t)\rangle = 1$. Since the speed of γ must be $f\nu$ we conclude

$$f = \lambda H^\alpha = \frac{1}{|Du|}$$

which implies the degenerate elliptic PDE

$$\operatorname{div}\left(\frac{Du}{|Du|}\right) = \left(\frac{1}{\lambda |Du|}\right)^{1/\alpha}.$$

Assume that $E_0 \subset \mathbb{R}^{n+1}$ is such that $M_0^n = \partial E_0$ and that $M_t^n \cap E_0 = \emptyset$. Then we can impose the boundary condition $u = 0$ on ∂E_0 and want to solve the above PDE outside of E_0, i.e.

$$(\star)\begin{cases} \operatorname{div}\left(\frac{Du}{|Du|}\right) = \left(\frac{1}{\lambda |Du|}\right)^{1/\alpha} & \text{in } \mathbb{R}^{n+1} \setminus \overline{E_0} \\ u = 0 & \text{on } \partial E_0. \end{cases}$$

If we consider hypersurfaces with boundary such that the hypersurfaces move along but stay perpendicular to a given supporting hypersurface with normal μ. Then the contact angle condition $\langle \nu, \mu\rangle = 0$ implies the Neumann condition $D_\mu u = 0$ on Σ. To keep things easier we will only consider closed hypersurfaces here.

9.2. Solving the level set problem

To deal with a degenerate elliptic PDE in an unbounded domain one can regularize the PDE and construct a bounded domain by introducing the set F_L which should contain E_0 and put

$$(\star)_\varepsilon \begin{cases} \operatorname{div}\left(\dfrac{Du^\varepsilon}{\sqrt{\varepsilon^2+|Du^\varepsilon|^2}}\right) = \lambda^{-1/\alpha}(\varepsilon^2+|Du^\varepsilon|^2)^{-1/2\alpha} & \text{in } F_L \setminus \overline{E_0} \\ u^\varepsilon = 0 & \text{on } \partial E_0 \\ u^\varepsilon = L & \text{on } \partial F_L. \end{cases}$$

The idea is that for $\varepsilon \to 0$ we send $L \to \infty$ and $F_L \to \mathbb{R}^{n+1}$ such that we recover $(\star)$ in the limit. This reduces the existence prove into three steps:

1. Prove the existence of smooth/ classical solutions u^ε of $(\star)_\varepsilon$.

2. Check how much regularity survives in the limit $u^\varepsilon \to u$ as $\varepsilon \to 0$.

3. Interpret the solution u geometrically, i.e. find a suitable notion of weak solutions.

Here is a very short outline of these three steps in the case of IMCF, i.e. $\lambda = 1$ and $\alpha = -1$:

Step 1: From what we have learned about quasilinear elliptic Dirichlet problems we know that the first step can be reduced to proving C^1-apriori estimates for u^ε, i.e. it suffices to control $|u^\varepsilon|$ and $|Du^\varepsilon|$. The gradient estimate can be obtained by constructing suitable barriers at the boundary and the existence of an interior maximum of the gradient can be excluded using the maximum principle.

Step 2: It turns out that the C^1-estimate can be obtained independently of ε. However, the full $C^{2,\alpha}$-estimates will depend on ε. Therefore, the limiting function u will only be in $C^{0,1}_{loc}(\mathbb{R}^{n+1} \setminus E_0)$. The convergence is uniform on compact subsets.

Step 3: It turns out that the limiting function u is the minimizer of a function. Namely,

$$J_u^K(u) \leq J_u^K(v) := \int_K (|Dv| + v|Du|)\,, \qquad \forall\, v \in C^{0,1}_{loc}(\mathbb{R}^{n+1} \setminus E_0),\ \{u \neq v\} \subset K$$

In order to prove that u minimizes this functional you need the following ingredients:

a) Compatibility: Classical solutions are weak solutions.

b) Compactness: If a sequence of weak solutions converges locally uniformly to a limit. Then this limit is a weak solution too.

Now assume that u^{ε_i} is a sequence of solutions of $(\star)_{\varepsilon_i}$ converging to u. Obviously, in one dimension higher the functions $U_i(x,z) := u^{\varepsilon_i} + \varepsilon_i z$ will converge to the function $U(x,z) := u(x)$. Furthermore, the miracle happens that U_i satisfies $(\star)$, i.e. classical IMCF in one dimension higher. By the compatibility those are also weak solutions and by the compactness also U is a weak solution. Finally, a cut-off argument shows that u

itself is a weak solution.

The fact that u minimizes a functional helps to prove further analytic and geometric properties of the solution. It helps to prove a regularity result for the M_t^n, the existence of mean curvature in L^∞ and the geometric property that the M_t^n are outward minimizing. In the case of the torus (which was an example where the classical flow breaks down in finite time) as a result of being outward minimizing the weak solution will change its topology and become spherical and continues to expand.

From the view point of mathematical physics the main feature of the weak solution is that it still keeps the Hawking mass

$$m_{Haw}(M_t^n) := \frac{|M_t^n|^{1/2}}{(16\pi)^{3/2}}\left(16\pi - \int_{M_t^n} H^2\, d\mu_t\right)$$

monotone. This is a main ingredient in the prove of the Riemannian Penrose inequality. For further detail about this subject see [5]. There Bray explains the proof by Huisken-Ilmanen which is based on the concept of weak solutions which we outlined above. Furthermore, he explains his prove which covers a bit more general setting of the Riemannian Penrose inequality. The full Penrose inequality is still an open problem.

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