

Problem set 2

1. Let $A \subset X$ be a non-empty subset and assume that $\tilde{H}_*(A) = 0$ (that is, A is acyclic). Prove that $H_*(X, A) \cong \tilde{H}_*(X)$.
2. Suppose that X is a path-connected space and let $f : X \rightarrow X$ be a map. Prove that the induced map $f_* : H_0(X) \rightarrow H_0(X)$ is the identity.
3. Let $f : (X, x_0) \rightarrow (Y, y_0)$ be a map of pointed spaces and consider the induced maps $f_\# : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$ and $f_* : H_1(X) \rightarrow H_1(Y)$. Prove commutativity of the diagram

$$\begin{array}{ccc}
 \pi_1(X, x_0) & \xrightarrow{f_\#} & \pi_1(Y, y_0) \\
 \downarrow \phi_X & & \downarrow \phi_Y \\
 H_1(X) & \xrightarrow{f_*} & H_1(Y)
 \end{array}$$

where ϕ_X and ϕ_Y are the Hurewicz homomorphisms.

4. Let $p : X \rightarrow Y$ be a covering map, and let $x_0 \in X$ and $y_0 = p(x_0)$. Prove that the map $\pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$ is a monomorphism. Is it true in general that $p_* : H_1(X) \rightarrow H_1(Y)$ is a monomorphism?
5. Let H be a “theory” satisfying axioms 1-4 of a homology theory, but not necessarily axiom 5. Show that

$$(i_X)_* \oplus (i_Y)_* : H_p(X) \oplus H_p(Y) \rightarrow H_p(X \sqcup Y)$$

is an isomorphism for all spaces X, Y and for all $p \in \mathbb{Z}$, where $i_X : X \hookrightarrow X \sqcup Y$, $i_Y : Y \hookrightarrow X \sqcup Y$ denote the inclusions into the disjoint union.

Hints.

- (a) Consider the long exact sequence of the pair $(X \sqcup Y, X)$.
- (b) Consider the excision $(Y, \emptyset) = ((X \sqcup Y) \setminus X, X \setminus X) \xrightarrow{k} (X \sqcup Y, X)$ and the resulting isomorphism $k_* : H_*(Y) \xrightarrow{\cong} H_*(X \sqcup Y, X)$.
- (c) Note that the following diagram commutes:

$$\begin{array}{ccc}
 (Y, \emptyset) & \xrightarrow{k} & (X \sqcup Y, X) \\
 \searrow i_Y & & \nearrow j \\
 & X \sqcup Y &
 \end{array}$$

(d) Deduce that in the long exact sequence

$$\cdots \rightarrow H_p(X) \xrightarrow{(i_X)_*} H_p(X \sqcup Y) \xrightarrow{j_*} H_p(X \sqcup Y, X) \rightarrow \cdots$$

all maps j_* are surjective, and that thus the sequence gives rise to short exact sequences

$$0 \rightarrow H_p(X) \xrightarrow{(i_X)_*} H_p(X \sqcup Y) \xrightarrow{j_*} H_p(X \sqcup Y, X) \rightarrow 0.$$

(e) Find a right inverse for j_* to show that these short exact sequences split.