Problem set 3

Notation. For a space X, $H_*(X)$ denotes its homology with coefficients in $\mathbb{Z}$.

1. Compute $H_*(\mathbb{R}P^n;\mathbb{Z}_2)$. Compare the result with $H_*(\mathbb{R}P^n)$ (which was discussed in class).

2. The 3-torus is the quotient space $T^3 = \mathbb{R}^3/\mathbb{Z}^3 \approx S^1 \times S^1 \times S^1$. Find a CW-structure on T^3 and use it to compute $H_*(T^3)$.

3. Consider the space X which is the union of the unit sphere $S^2 \subset \mathbb{R}^3$ and the line segment between the north and south poles (cf. problem 1.5).

   (a) Give X a CW-structure and use it to compute $H_*(X)$.

   (b) Use that X is homotopy equivalent to $S^2 \vee S^1$ to give an easier computation of $H_*(X)$.

4. Let C be the circle on the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ which is the image, under the covering map $\mathbb{R}^2 \to T^2$, of the line $px = qy$. Define $X = T^2/C$, the quotient space obtained by identifying C to a point. Compute $H_*(X)$.

5. Show that the quotient map $S^1 \times S^1 \to S^2$ collapsing the subspace $S^1 \vee S^1 \subset S^1 \times S^1$ to a point is not null-homotopic by showing that it induces an isomorphism on H_2. On the other hand, show via covering spaces that any map $S^2 \to S^1 \times S^1$ is null-homotopic.

6. Compute $H_*(\mathbb{R}P^n/\mathbb{R}P^m)$ for $m < n$, using cellular homology and equipping $\mathbb{R}P^n$ with the standard CW-structure with $\mathbb{R}P^m$ as its m-skeleton.