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Algebraic Topology I

HS 2013

Problem set 5

1. Recall that a retraction of a space X onto a subspace $A \subset X$ is a map $r : X \to A$ such that $r|_A = \mathrm{id}_A$. Prove that there exists no retraction $f : D^n \to \partial D^n$.

2. Prove the Brouwer fixed point theorem for D^n, which says that every continuous map $f : D^n \to D^n$ has a fixed point.

3. Prove the fundamental theorem of algebra, i.e., that every polynomial function $p : \mathbb{C} \to \mathbb{C}$ of degree $d \geq 1$ has at least one zero. *Hint:* You might think about what p does suitably to chosen circles in $\mathbb{C}$, and about degrees.

4. Prove that there does not exist a nowhere-vanishing vector field on any even-dimensional sphere S^{2n}.

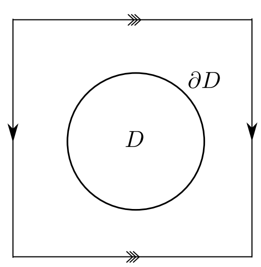
5. Prove that there exists a surjective map $S^n \to S^n$ of degree zero for any $n \geq 1$.

6. Construct a degree k map $S^n \to S^n$ for every $k \in \mathbb{Z}$ and every $n \geq 1$.

7. Let $f : \mathbb{R}P^2 \to \mathbb{R}P^2$ be a map which induces an isomorphism in homology. Prove that f is surjective.

8. Let $f : (D^n, S^{n-1}) \to (D^n, S^{n-1})$ be a map such that $\deg f|_{S^{n-1}} \neq 0$. Prove that f is surjective.

9. Consider the torus T^2 and an embedded disc $D \subset T^2$ as in the picture. Set $X = T^2 \setminus \mathrm{int}\, D$. Prove that X does not retract onto $\partial D \subset X$.



10. Prove that any cycle c that represents a non-trivial class in $H_n(S^n)$ must cover all of S^n (i.e., the union of the images of all simplices constituting c is all of S^n).

11. Find a CW complex structure for the closed orientable genus g surface Σ_g and compute $H_*(\Sigma_g)$ using cellular homology. *Hint:* Recall from problem set 4 the construction of Σ_g from a $4g$-sided polygon.

12. Let X be a finite CW complex and suppose that $p : \widetilde{X} \to X$ is an n-sheeted covering map. Prove that $\chi(\widetilde{X}) = n\chi(X)$.

13. Show that if the closed orientable surface Σ_g is a covering space of Σ_h, then $g = n(h-1)+1$ for some n.

14. Suppose we build S^2 from a finite collection of polygons by identifying edges in pairs. Show that in the resulting CW complex structure on S^2, the 1-skeleton cannot be either of the following graphs: