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Solutions to problem set 1

Notation. $I := [0,1]$; we omit $\circ$ in compositions: $fg := f \circ g$.

1. It follows straight from the definitions of "mapping cone" and "mapping cylinder" that the map $\tilde{F} : M_{f\phi} \to M_f$ given by $y \mapsto y$ for $y \in Y$ and $(x', t) \mapsto (\phi(x'), t)$ for $(x', t) \in X' \times I$ is well-defined and descends to a map $F : C_{f\phi} \to C_f$. We will show that $\tilde{F}$ is a homotopy equivalence, which implies that the same is true for F.

   Let $\psi : X \to X'$ be a homotopy inverse of ϕ, and let $g : X \times I \to X$ be a homotopy connecting $\phi\psi$ to id_X. Consider the map $\tilde{G} : M_{f\phi\psi} \to M_{f\phi}$ defined analogously to $\tilde{F}$; that is, it maps $y \mapsto y$ and $(x, t) \mapsto (\psi(x'), t)$.

   The composition $\tilde{F}\tilde{G} : M_{f\phi\psi} \to M_f$ is then given by $y \mapsto y$ and $(x, t) \mapsto (\phi\psi(x), t)$. Note that we have another map $h : M_{f\phi\psi} \to M_f$ induced by the homotopy $fg : X \times I \to Y$ connecting $f\phi\psi$ to f: h is given by $y \mapsto y$ and

$$(x,t) \mapsto \begin{cases} fg(x, 2t) \in Y & t \leq \frac{1}{2} \\ (x, 2t-1) \in X \times I & \frac{1}{2} \leq t \end{cases}$$

   and [Bredon, Theorem I.14.18] tells us that h is a homotopy equivalence. We will now show that $\tilde{F}\tilde{G} \simeq h$. Consider the homotopy $\Sigma : M_{f\phi\psi} \times I \to M_f$ given by $(y, s) \mapsto y$ for $(y, s) \in Y \times I$ and

$$(x,t,s) \mapsto \begin{cases} fg(x, 2t) \in Y & t \leq \frac{s}{2} \\ \left(g(x,s), \frac{2}{2-s}t - \frac{s}{2-s}\right) \in X \times I & \frac{s}{2} \leq t \end{cases}$$

   for $(x, t, s) \in (X \times I) \times I$. Note that Σ is well-defined and continuous and satisfies $\Sigma(\cdot, 0) = \tilde{F}\tilde{G}$ and $\Sigma(\cdot, 1) = h$. It follows that $\tilde{F}\tilde{G}$ is a homotopy equivalence, too.

   In an analogous manner, one can define a map $\tilde{H} : M_{f\phi\psi\phi} \to M_{f\phi\psi}$ and then show that $\tilde{G}\tilde{H} : M_{f\phi\psi\phi} \to M_{f\phi\psi}$ is a homotopy equivalence.

   The rest of the proof is analogous to the end of the proof of [Bredon, Theorem I.14.19]. Namely, let k and k' be homotopy inverses of $\tilde{F}\tilde{G}$ resp. $\tilde{G}\tilde{H}$, so that in particular $k\tilde{F}\tilde{G} \simeq \mathrm{id}_{M_{f\phi\psi}}$ and $\tilde{G}\tilde{H}k' \simeq \mathrm{id}_{M_f}$. Then $L := k\tilde{F}$ and $R := \tilde{H}k'$ are left resp. right homotopy inverses of $\tilde{G}$. By Problem 4 on this sheet, L and R are in fact homotopy inverses of $\tilde{G}$, and hence $\tilde{G}$ is a homotopy equivalence.

   Hence $\tilde{F}\tilde{G}$ and $\tilde{G}$ are both homotopy equivalences, from which it follows that also $\tilde{F}$ is a homotopy equivalence.

2. By [Bredon, Prop. I.14.5], contractibility of X means that id_X is homotopic to a map whose image is a singleton $\{x_0\} \in X$; that is, there exists a map $h : X \times I \to X$ such that $h(x, 0) = x$ and $h(x, 1) = x_0$ for all $x \in X$. Consider now the map $h' = r \circ h : A \times I \to A$. For $a \in A$, it satisfies $h'(a, 0) = r(h(a, 0)) = r(a) = a$ by the defining property of retraction, and $h'(a, 1) = r(x_0)$; in other words, id_A is homotopic to a map with image $\{r(x_0)\} \subset A$, and hence A is contractible (again by [Bredon, Prop. I.14.5]).

3. Recall that $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$. We claim that the expression

$$(x,t) \mapsto \frac{(1-t)f(x) + tg(x)}{\|(1-t)f(x) + tg(x)\|},$$

yields a well-defined map $\phi : X \times I \to S^n$. Pretending that this is true, note that $\phi(x,0) = f(x)$ and $\phi(x,1) = g(x)$ for all $x \in X$, and hence f and g are homotopic.

To prove well-definedness, we must show that the denominator never vanishes. To do so, we assume the contrary, i.e., that there exists some $(x,t) \in X \times I$ such that $(1-t)f(x)+tg(x) = 0$. This is equivalent to $(1-t)f(x) = -tg(x)$, from which we obtain $(1-t)\cdot\|f(x)\| = t\cdot\|g(x)\|$; since $\|f(x)\| = \|g(x)\| = 1$, it follows that $1-t = t$, and hence $t = \frac{1}{2}$. Inserting this into the first equation, we obtain $f(x) = -g(x)$, which contradicts our assumption.

4. Using the assumptions $fg \simeq \mathrm{id}_Y$ and $hf \simeq \mathrm{id}_X$, we obtain

$$fh \;=\; fh\,\mathrm{id}_Y \;\simeq\; fhfg \;\simeq\; f\,\mathrm{id}_X\, g \;=\; fg \;\simeq\; \mathrm{id}_Y.$$

Hence h is in fact a homotopy inverse of f, and thus f is a homotopy equivalence. (Similarly, one can show that g is a homotopy inverse of f.)

5. One could prove these statements by writing down explicit maps and homotopies. To facilitate this, one could use, for example, the following "model" for S^2 (and similar models for $X, V, S^2 \vee S^1$ and $S^2 \vee S^2$). The map

$$S^1 \times [-1,1] \to S^2, \quad (x_1, x_2, h) \mapsto \big(\sqrt{1-h^2}\,x_1, \sqrt{1-h^2}\,x_2, h\big),$$

induces a homeomorphism

$$(S^1 \times [-1,1])/\sim \quad \approx \quad S^2,$$

where $\sim$ is the equivalence relation that identifies all points of $S^1 \times \{-1\}$ with each other, and all points of $S^1 \times \{+1\}$ with each other. (Namely, the map is continuous and bijective, and since $S^1 \times [-1,1]$ is compact, it is a homeomorphism.)

Another less direct way of proving the required statements is to use the following fact: *If (Z, A) is a CW pair consisting of a CW complex Z and a contractible subcomplex A, then the quotient map $Z \to X/A$ is a homotopy equivalence.* (See [Hatcher p.11].) To use this, one must endow the spaces X and Y with appropriate structures of CW complexes.

For X, consider an additional arc A on S^2 connecting the north and the south pole; a possible CW complex structure has the two poles as 0-cells, the interiors of the "stick" and the arc A as 1-cells, and the rest of S^2 as a 2-cell. The statement above implies that $X \to X/A$ is a homotopy equivalence, but also that X/A is homotopy equivalent to $S^2 \vee S^1$ (namely, denote by $\tilde{X}$ the CW complex obtained by removing the stick, whose associated space is S^2; then $\tilde{X}/A$ is homotopy equivalent to S^2 by the statement above; from this it follows easily that X/A is homotopy equivalent to $S^2 \vee S^1$).

For Y, consider the CW complex structure with one 0-cell on the equator, one 1-cell (the rest of the equator), and three 2-cells (northern and southern hemispheres and the disc in the equator plane). Denoting the last-mentioned 2-cell by A, the statement above tells us that $Y \to Y/A$ is a homotopy equivalence; on the other hand, it is easy to see (using e.g. the model above), that Y/A is homeomorphic to $S^2 \vee S^2$.

6. We view $\mathbb{R}P^2$ as $D^2/\sim$, where $\sim$ is the equivalence relation that identifies antipodal points on the boundary. That is, $x, y \in D^2$ satisfy $x \sim y$ if and only if $x = y$, or x and y both lie on $S^1 \subset D^2$ and satisfy $x = -y$.

Consider the map $F : S^1 \sqcup (S^1 \times I) \to \mathbb{R}P^2$ defined on S^1 by $e^{2\pi i t} \mapsto [e^{\pi i t}]$ for $t \in [0,1]$, and on $S^1 \times I$ by $(e^{2\pi i t}, s) \mapsto [(1-s)e^{2\pi i t}]$. (Note that the first part is well-defined and continuous, because $e^{2\pi i 0} = -1 \sim 1 = e^{2\pi i 1}$ and hence $[e^{2\pi i 0}] = [e^{2\pi i 1}] \in \mathbb{R}P^2$.)

Observe that F descends to a well-defined map $F' : M_f \to \mathbb{R}P^2$, because $F(f(e^{2\pi i t})) = F(e^{4\pi i t}) = [e^{2\pi i t}] = F(e^{2\pi i t}, 0)$, and moreover F' is clearly surjective. Since F' maps all of $S^1 \times \{1\}$ to a single point, namely $[0] \in \mathbb{R}P^2$, it descends further to a map $F'' : C_f \to \mathbb{R}P^2$. Note that F'' is bijective, since F' is injective on $M_f \setminus (S^1 \times \{1\})$.

Since C_f is compact and $\mathbb{R}P^2$ is Hausdorff, we conclude using [Bredon, Theorem I.7.8] that ϕ is a homeomorphism.