Algebra I

Serie 10

Representation theory

1. Let η be a one-dimensional character of G. Show that for any irreducible representation V of G, the representation $V \otimes \eta$ is also irreducible.

2. Determine the character table of S_4.

   (a) What are the conjugacy classes of S_4 ?

   (b) Let $V = \mathbb{C}^4$ be the permutation representation associated to the action of S_4 on the canonical coordinates of $\mathbb{C}^4$, i.e. $\sigma.e_i = e_{\sigma(i)}$ for every $\sigma \in S_4$ and every $i \in \{1, \ldots, 4\}$. Compute χ_V and $\langle \chi_V, \chi_V \rangle$. Conclude that V is the direct sum $V = V_1 \oplus V_2$ of two irreducible representations V_1 and V_2 that are not isomorphic.

   (c) Show that the hyperplane W defined by $\sum_{i=1}^4 x_i = 0$ is S_4-invariant. (Hint : Consider the difference $v - \sigma.v$, when $\sigma \in S_4$ is a transposition.) Conclude that W is irreducible. How can you compute χ_W ?

   (d) The signature homomorphism sign is a one-dimensional character of S_4. Show that W is not isomorphic to $W \otimes$ sign.

   (e) How can you complete the character table of S_4 ? (Hint : Consider the character of the regular representation.)

3. Let G be a finite group.

   (a) Let χ be the character of a representation of G. Show that

   $$K_\chi = \{g \in G : \chi(g) = \chi(1)\}$$

   is a normal subgroup of G.

   (b) Prove that G is simple if and only if $K_\chi = \{1\}$, for every non-trivial irreducible character χ. How can you read the property of being simple off the character table ?

4. Let G be an abelian group (not necessarily finite !). Show that any irreducible finite-dimensional representation of G has dimension 1. (Hint : Schur's Lemma.)

5.* Let G_1 and G_2 be finite groups with representations V_1 and V_2 (respectively). The direct product $G = G_1 \times G_2$ has a representation $V_1 \boxtimes V_2$ given by the vector space $V_1 \otimes V_2$ and the action $\rho(g_1, g_2)(x \otimes y) = \rho_{V_1}(g_1)(x) \otimes \rho_{V_2}(g_2)(y)$.

(a) Show that if V_1 and V_2 are irreducible representations, then $V_1 \boxtimes V_2$ is an irreducible representation of G. (Hint : You can use the fact that $\mathrm{tr}(\mathrm{u}_1 \otimes \mathrm{u}_2) = \mathrm{tr}(\mathrm{u}_1)\mathrm{tr}(\mathrm{u}_2)$ for endomorphisms $u_1 \in \mathrm{End}(\mathrm{V}_1)$, $u_2 \in \mathrm{End}(\mathrm{V}_2)$. Then compute the character of $V_1 \boxtimes V_2$ from that of V_1 and V_2.)

(b) Show that any irreducible representation of G is obtained in this way.