Exercise 11

Rings: definitions, units, zero divisors, polynomial rings

1. Show that the matrices $M(n \times n, \mathbb{C})$ form a noncommutative ring. What are the units of $M(n \times n, \mathbb{C})$? What are the zero divisors?

2. Let us set $\alpha = \sqrt{2}$, $\beta = \sqrt{3}$, $\gamma = \alpha + \beta$.
   (a) Show that γ is an algebraic number.
   (b) Is $\mathbb{Q}[\alpha, \beta] = \mathbb{Q}[\gamma]$ as subrings of $\mathbb{R}$?
   (c) Is $\mathbb{Z}[\alpha, \beta] = \mathbb{Z}[\gamma]$? *Hint:* you can use the equation determined in (a) to show that every element of $\mathbb{Z}[\gamma]$ can be written as an integer combination of $1, \gamma, \gamma^2, \gamma^3$.

3. (a) Let R be a finite ring. Show that every $x \in R$ is either 0, a zero divisor, or invertible.
   (b) Give an example of a ring R that contains an element that is not a zero divisor and is not invertible.
   (c) Give an example of an infinite ring that is not an integral domain.

4. Let R_1 and R_2 be rings, show that $R_1 \times R_2$ is a ring. Show that if $R_1 \times R_2$ is an integral domain, then at least one of the two rings R_1, R_2 is the zero ring.

5. For the following (commutative) rings determine the zero divisors and the units, say if they are integral domains:
   (a) $\mathbb{Z}/m\mathbb{Z}$ with the product induced by the product of $\mathbb{Z}$.
   (b) the set $F = C([0,1], \mathbb{R})$ of continuous functions of the interval $[0,1]$ with pointwise addition and multiplication.
   (c) The subring R of $M(2 \times 2, \mathbb{R})$ consisting of matrices of the form $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$, with a, b in $\mathbb{R}$.

6. Let R be a (commutative) ring, and let us consider the polynomial ring $R[x]$.
   (a) Show that if R is an integral domain then the same is true for $R[x]$.
   (b) Prove that if R is an integral domain, then the set of units of $R[x]$ coincides with the set of units of R.
   (c) Show that $1 + 5X$ is invertible in $\mathbb{Z}/25\mathbb{Z}[x]$.

7. For which positive n does $x^2 + x + 1$ divide $x^4 + 3x^3 + x^2 + 7x + 5$ in $\mathbb{Z}/n\mathbb{Z}[x]$?