1. Show that the image of a representation of dimension 1 of a finite group is a cyclic group.

**Solution** Let $G$ be a finite group. A representation of dimension 1 of $G$ is an homomorphism $\rho : G \to GL_1(\mathbb{C})$. The group $GL_1(\mathbb{C})$ coincides with the multiplicative group $\mathbb{C}^\times$. Let $z$ be an element in the image of $\rho$. Since $G$ is finite, we get that $z^n = 1$ for some $n$, in particular the image of $\rho$ is a subgroup of the group $S^1$ of complex numbers of modulus one. We saw in class that all finite subgroups of $S^1$ are cyclic and this finishes the proof.

2. Let $H$ be a subgroup of index 2 of a group $G$, and let $\sigma : H \to GL(V)$ be a representation. Let $a$ be an element in $G$ not in $H$. Define a conjugate representation $\sigma' : H \to GL(V)$ by the rule $\sigma'(h) = \sigma(a^{-1}ha)$. Prove that

(a) $\sigma'$ is a representation of $H$.

**Solution** In order to check that $\sigma'$ is well defined, let us notice that, since $H$ has index 2 in $G$, $H$ is normal, hence for every element $h \in H$ the element $a^{-1}ha$ still belongs to $H$, in particular the value $\sigma(a^{-1}ha)$ is well defined. Let us now check that $\sigma'$ is a representation. In order to do this, it is enough to verify that $\sigma'$ is an homomorphism in $GL_n$ for some $n$. In particular, since $\sigma$ is a representation in $GL(V)$, the image of $\sigma'$ is also contained in $GL(V)$, hence we only have to verify that $\sigma'(gh) = \sigma'(g)\sigma'(h)$. But this follows from the definition:

$$\sigma'(gh) = \sigma(a^{-1}gha) = \sigma((a^{-1}ga)(a^{-1}ha)) = \sigma(a^{-1}ga)\sigma(a^{-1}ha) = \sigma'(g)\sigma'(h).$$

(b) If $\sigma$ is the restriction to $H$ of a representation of $G$ then $\sigma'$ is isomorphic to $\sigma$.

**Solution** Assume that $\sigma$ is the restriction of a representation of $G$, and let $A \in GL(V)$ be the element $\sigma(a)$. In order to show that $\sigma$ and $\sigma'$ are conjugate we need a linear isomorphism $L : V \to V$ such that, for any $h \in H$, $\sigma(h)(Lv) = L(\sigma'(h)v)$. The linear map $A$ is such an isomorphism, indeed for any $v \in V$:

$$\sigma(h)Av = AA^{-1}\sigma(h)Av = A\sigma(a^{-1})\sigma(h)\sigma(a)v = A\sigma(a^{-1}ha) = A\sigma(h)v.$$

(c) If $b$ is another element of $G$ not in $H$, then the representation $\sigma'' = \sigma(b^{-1}hb)$ is isomorphic to $\sigma'$.

**Solution** Since $b$ is another element of $G$ that doesn’t belong to $H$, and since
$H$ has index 2 in $G$, then there exists an element $h$ in $H$ such that $b = ah$. Now we have, for every $g \in H$, that $\sigma''(g) = \sigma(h^{-1}a^{-1}gah) = \sigma(h^{-1})\sigma'(g)\sigma(h)$. In particular this implies that, for every $g \in H$, we have $\sigma(h)\sigma''(g) = \sigma'(g)\sigma(h)$ and the linear map $\sigma(h) : V \rightarrow V$ gives an isomorphism of the representations $\sigma'$ and $\sigma''$.

3. Let $\rho : G \rightarrow \text{GL}(V)$ be a representation of a finite group on a real vector space $V$. Prove the following:

(a) There exists a $G$-invariant, positive definite, symmetric form $\langle , \rangle$ on $V$.

**Solution** Let us fix a positive definite, symmetric bilinear form $[ , ]$ on $V$. (To find such a form it is enough to fix an isomorphism of $V$ with $\mathbb{R}^n$ and consider the standard positive definite symmetric bilinear form on $\mathbb{R}^n$). And let us define the averaged form by setting

$$\langle v, w \rangle = \frac{1}{|G|} \sum_{g \in G} [\rho(g)v, \rho(g)w].$$

The form is symmetric, positive definite and $G$-invariant. The fact that is symmetric follows from the fact that $[ , ]$ is symmetric:

$$\langle v, w \rangle = \frac{1}{|G|} \sum_{g \in G} [\rho(g)v, \rho(g)w] = \frac{1}{|G|} \sum_{g \in G} [\rho(g)w, \rho(g)v] = \langle v, w \rangle.$$

To check that the form is positive, it is enough to check that $\langle v, v \rangle > 0$, but we have

$$\langle v, v \rangle = \frac{1}{|G|} \sum_{g \in G} [\rho(g)v, \rho(g)v] > 0$$

since the latter expression is a sum of positive numbers, the verification of the $G$ invariance follows rearranging the summation, once one notices that for any element $h \in G$, the right multiplication by $h$ gives a permutation of $G$:

$$\langle \rho(h)v, \rho(h)w \rangle = \frac{1}{|G|} \sum_{g \in G} [\rho(g)\rho(h)v, \rho(g)\rho(h)w]$$

$$= \frac{1}{|G|} \sum_{gh \in G} [\rho(gh)v, \rho(gh)w] = \langle v, w \rangle.$$

(b) $\rho$ is a direct sum of irreducible representations.

**Solution** If $\rho$ is irreducible, there is nothing to prove. Assume that $\rho$ is not irreducible and let $W < V$ a $\rho$-invariant subspace. We claim that the orthogonal of $W$ with respect to the form $\langle , \rangle$ defined in the previous part is also $\rho$-invariant. Indeed it is enough to check that if $z \in W^\perp$ and $g \in G$, then $\rho(g)z$ is in $W^\perp$, or equivalently for every $w \in W$ $\langle \rho(g)z, w \rangle = 0$. But this is true since

$$\langle \rho(g)z, w \rangle = \langle \rho(g^{-1})\rho(g)z, \rho(g^{-1})w \rangle = \langle z, \rho(g^{-1})w \rangle = 0$$
Here the first equality is due to the fact that \( \langle , \rangle \) is \( \rho \)-invariant, the second one to the fact that \( \rho \) is a representation, the third to the fact that \( \rho(g^{-1})w \in W \) since \( W \) is \( G \)-invariant, and \( z \) belongs to \( W^\perp \). This implies that \( \rho \) splits as a direct sum of two representations \( \rho', \rho'' \). Since \( V \) is finite dimensional the conclusion follows by induction.

4. Let us consider the representation \( \rho \) of \( \mathbb{Z} \) on \( \mathbb{C}^2 \) defined by \( \rho(1) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \).

(a) Find a proper invariant subspace.

**Solution** Clearly the subspace of \( \mathbb{C}^2 \) generated by the first element of the standard basis is invariant under the representation \( \rho \): indeed any element in \( \langle e_1 \rangle \) has expression \( [a] \) for some \( a \in \mathbb{C} \), and for any \( n \in \mathbb{Z} \), we have that \( \rho(n) \) is the matrix \( \begin{bmatrix} 1 & n \\ 1 & 1 \end{bmatrix} \) and we have \( \begin{bmatrix} 1 & n \\ 1 & 1 \end{bmatrix} [a] = [a] \).

(b) Show that \( \rho \) is not the direct sum of irreducible representations.

**Solution** Assume by contradiction that \( \rho \) is the direct sum of irreducible representations. Then there would exist another vector \( w \in \mathbb{C}^2 \) that doesn’t belong to \( \langle e_1 \rangle \), and such that, for every \( n \in \mathbb{Z} \), \( \rho(n)w = w \). But since already \( \rho(1) \) has no other eigenvector apart from \( e_1 \) this is clearly not possible.

5. Determine the character table for the Klein four group.

**Solution** The Klein four group \( K \) is isomorphic to the product \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \). Since it is abelian its conjugacy classes coincides with the four group elements. Let us call \( a \) and \( b \) the generators of \( K \). In order to compute the character table for \( K \) it is enough to determine four distinct non isomorphic one dimensional representations of \( K \). If \( \rho \) is a one dimensional representation of \( K \), the image of a generator of \( K \) should be an element of order two in \( \mathbb{C}^\times \), in particular it should be an element of the set \( \{ \pm 1 \} \). Moreover for any choice we get a non isomorphic representation of \( K \). This implies that the character table of \( K \) is

\[
\begin{array}{cccc}
   & 1 & a & b & ab \\
\rho_1 & 1 & 1 & 1 & 1 \\
\rho_a & 1 & -1 & 1 & -1 \\
\rho_b & 1 & 1 & -1 & -1 \\
\rho_{ab} & 1 & -1 & -1 & 1 \\
\end{array}
\]

6. Let us consider the Dihedral group \( D_5 \), and its cyclic subgroup \( C_5 \).

(a) Determine the character table of \( D_5 \) and of \( C_5 \).

**Solution** Let \( \omega \) denote a primitive fifth root of unit in \( \mathbb{C}^\times \) that is a number such that \( \omega^5 = 1 \). There are five conjugacy classes in \( C_5 \), corresponding to the five elements. Moreover in order to determine a representation of \( C_5 \) it is enough to describe the image of the generator that is going to be a fifth root of unity, hence a power of \( \omega \). In particular we get that the character table of \( \mathbb{C}^5 \) is
In order to compute the conjugacy classes in $D_5$ let us notice that, since 5 is odd, all reflections are conjugate, hence form a conjugacy class $C_y$, moreover the rotations come in three different conjugacy classes: $\{1\}, \{x, x^4\}, \{x^2, x^3\}$. This implies that we have to exhibit five different irreducible representations of $D_5$. We will denote by $\rho_1$ the trivial representation. Let us consider the subgroup $C_5$ of $D_5$. It is a normal subgroup and the quotient $D_5/C_5 = \mathbb{Z}/2\mathbb{Z}$. This gives another one dimensional (hence irreducible) representation of $D_5$, the sign representation. We will denote it by sign.

Let us now consider the standard representation of $D_5$ as a subgroup of $O_2(\mathbb{R})$. We showed in class that the group $D_5$ is isomorphic to the subgroup of $O_2(\mathbb{R})$ generated by a reflection and a rotation of angle $2\pi/5$. The matrix expression for a rotation of angle $2\pi/5$ is $R_x = \begin{bmatrix} \cos(2\pi/5) & \sin(2\pi/5) \\ -\sin(2\pi/5) & \cos(2\pi/5) \end{bmatrix}$ and the matrix expression for the reflection along the $x$ axis is $R_y = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Interpreting these two matrices as complex matrices, and letting them act on $\mathbb{C}^2$ we get a complex representation of $D_5$ that we will denote by $\rho_\omega$. It is well known that if $\omega$ can be chosen to be $\cos(2\pi/5) + i \sin(2\pi/5)$. In particular it is easy to compute the character of $\rho_\omega$ (see the table below). Since the character has norm one, we get that the representation is irreducible. The last irreducible representation $\rho_{\omega^2}$ of $D_5$ is obtained in a similar manner, by defining $\rho_{\omega^2}(y) = R_y$ and $\rho_{\omega^2}(x) = R_x^2$. Computing the character of this representation one can easily check that $\rho_{\omega^2}$ is irreducible and $\rho_{\omega^2}$ is not isomorphic to $\rho_\omega$. This leads to the character table for $D_5$.

<table>
<thead>
<tr>
<th>$D_5$</th>
<th>${1}$</th>
<th>${x, x^4}$</th>
<th>${x^2, x^3}$</th>
<th>$C_y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>sign</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$\rho_\omega$</td>
<td>2</td>
<td>$\omega + \overline{\omega}$</td>
<td>$\omega^2 + \overline{\omega^2}$</td>
<td>0</td>
</tr>
<tr>
<td>$\rho_{\omega^2}$</td>
<td>2</td>
<td>$\omega^2 + \overline{\omega^2}$</td>
<td>$\omega + \overline{\omega}$</td>
<td>0</td>
</tr>
</tbody>
</table>

(b) Decompose the restriction of each irreducible character of $D_5$ into irreducible characters of $C_5$.

**Solution** The restriction to $C_5$ of the characters of $\rho_1$ and sign equal to the trivial character, the character of $\rho_\omega$ is the sum of the characters of $\tau_\omega$ and $\tau_{\omega^4}$, in a similar way $\rho_{\omega^2}$ is the direct sum of $\tau_{\omega^2}$ and $\tau_{\omega^3}$.

7. The Quaternion group $Q$ is the group $Q = \langle i, j, k \mid i^2 = j^2 = k^2 = -1, ijk = -1 \rangle$
(a) Find a subgroup of $GL_2(\mathbb{C})$ isomorphic to $Q$, determine the order of $Q$.

**Solution** Let us consider the elements of $GL_2(\mathbb{C})$ $I = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $K = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$. We have $I^2 = -\text{Id}$, $J^2 = -\text{Id}$, $K^2 = -\text{Id}$, moreover $IKJ = \begin{bmatrix} i & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = -\text{Id}$. In particular the subgroup $Q$ can be realized as a subgroup of $GL_2(\mathbb{C})$ and has eight elements: $\{\pm \text{Id}, \pm I, \pm J, \pm K\}$.

(b) Determine the conjugacy classes of $Q$.

**Solution** We will now identify $Q$ with the subgroup of $GL_2(\mathbb{C})$ we just defined, to make explicit computations. Since the matrices $\pm \text{Id}$ commute with every matrix in $GL_2(\mathbb{C})$, in particular they commute with the elements in $Q$, hence they are in the center of $Q$. Moreover, from the fact that $I^2 = -1$ we get that $I^{-1} = -I$. We can now compute the relation $IJ^{-1} = -[i 0] [0 1] [i 0] = -J$. Analogously one gets that $KJ^{-1} = -J$, in particular the conjugacy class of $J$ contains the two elements $\pm J$. In the same way one checks that the conjugacy class of $I$ contains $\pm I$ and the conjugacy class of $K$ contains $\pm K$.

(c) Prove that any subgroup of $Q$ is normal.

**Solution** The computation above shows that the subgroup generated by $I$ is isomorphic to $\mathbb{Z}/4\mathbb{Z}$ and is normal and the same is true for $\langle J \rangle$ and $\langle K \rangle$. The only other non trivial subgroup is the center $\{\pm 1\}$ of $Q$ and clearly the center is normal.

(d) Write the character table of $Q$.

**Solution** In order to compute the character table we need to find 5 irreducible representations of $Q$, since there are 5 conjugacy classes. Of course there is the trivial representation, that we will denote by $\rho_1$. Moreover we saw above that the subgroup generated by $I$ is normal. Since the quotient $Q/I$ is $\mathbb{Z}/2\mathbb{Z}$ we get a representation $\rho_i$ obtained by composing the sign representation of $\mathbb{Z}/2\mathbb{Z}$ with the quotient map. In the very same way (quotienting the subgroup generated respectively by $J$ and $K$) one gets the representations $\rho_j$ and $\rho_k$. All the representations $\rho_1$, $\rho_i$, $\rho_j$ and $\rho_k$ are one dimensional, hence irreducible, moreover computing the characters it is easy to see that they are not isomorphic, since the characters are different. We realized $Q$ as a subgroup of $GL_2(\mathbb{C})$. This gives a two dimensional representation $\rho$ of $Q$. That is irreducible since the eigenspaces of $I$ and $J$ are distinct. We can sum up the results we just obtained in the character table for $Q$:

<table>
<thead>
<tr>
<th>$Q$</th>
<th>${1}$</th>
<th>${-1}$</th>
<th>${\pm i}$</th>
<th>${\pm j}$</th>
<th>${\pm k}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\rho_i$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$\rho_j$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$\rho_k$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$\rho$</td>
<td>2</td>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>