Commutative Algebra

D-MATH, HS 2013

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Exercise Sheet 1

All exercises are taken from chapter 1 of the book *Introduction to Commutative Algebra* by Atiyah and Macdonald.

1. Let x be a nilpotent element of a ring A. Show that $1 + x$ is a unit of A. Conclude that the sum of a nilpotent element and a unit is a unit.

2. Let A be a ring and let $A[x]$ be the ring of polynomials in x with coefficients in A. Let $f = a_0 + a_1 x + \cdots + a_n x^n \in A[x]$. Show that

i) f is a unit in $A[x] \iff a_0$ is a unit in A and $a_1, \ldots, a_n$ are nilpotent.

ii) f is nilpotent $\iff a_0, \ldots, a_n$ is nilpotent.

iii) f is a zero-divisor $\iff \exists a \neq 0$ in A such that $af = 0$.

iv) f is primitive if the ideal general by $a_0, a_1, \ldots, a_n$ is equal to A, i.e. $(a_0, \ldots, a_n) = (1)$. Prove that if $f, g \in A[x]$, then fg is primitive $\iff$ f and g are primitive.

3. Let A be a ring $\neq 0$. Show that the set of prime ideals of A has minimal elements with respect to inclusion.

4. A ring A is called *Boolean* if $x^2 = x$ for all $x \in A$. In a Boolean ring A show that

(i) $2x = 0$ for all $x \in A$;

(ii) every prime ideal $\mathfrak{p}$ is maximal and $A/\mathfrak{p}$ is a field with two elements;

(iii) every finitely generated ideal in A is principal.

5. In a ring A, let Σ be the set of all ideals in which every element is a zero-divisor. Show that Σ has maximal elements and that every maximal element of Σ is a prime ideal. Conclude that the set of zero-divisors in A is a union of prime ideals.

6. Let A be a ring and let X be the set of all prime ideals of A. For each subset $E \subset A$, define

$$V(E) = \{\mathfrak{p} \in X \mid E \subset \mathfrak{p}\}.$$

Prove that the sets $V(E)$ satisfy the axioms for closed sets in a topological space, namely prove that:

i) $V(0) = X$ and $V(1) = \varnothing$ where $V(f) := V(\{f\})$.

ii) If $E_i, i \in I$ is any family of subsets of A, then

$$V\Big(\bigcup_{i \in I} E_i\Big) = \bigcap_{i \in I} V(E_i)$$

iii) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$ for ideals $\mathfrak{a}, \mathfrak{b}$ in A.

The resulting topology is called the *Zariski topology.* The set X equipped with the Zariski topology is called the *(prime) spectrum* of A, and is written $\operatorname{Spec}(A)$.

At last, show that if $\mathfrak{a}$ is an ideal in A and $r(\mathfrak{a})$ its radical ideal, then $V(\mathfrak{a}) = V(r(\mathfrak{a}))$

Due on Tuesday, Oct. 3, 2013