Commutative Algebra

D-MATH, HS 2013

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Exercise Sheet 10

Exercises **1. - 5.** are taken from the book *Introduction to Commutative Algebra* by Atiyah and Macdonald.

1. Let $f : A \longrightarrow B$ be a ring homomorphism and $f^* : \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$ the induced mapping. Show that

(i) Every prime ideal of A is a contracted ideal $\Longleftrightarrow$ f^* is surjective.

(ii) Every prime ideal of B is an extended ideal $\Longrightarrow$ f^* is injective.

Is the converse to (ii) also true? If not, can you give a counterexample?

2. Let A be a ring and let $X = \mathrm{Spec}(A)$ be its spectrum. Recall that for $f \in A$, we have the associated basic open set X_f (exercise sheet 2). Show that if $f, g \in A$, then

$$X_g \subset X_f \quad \Longleftrightarrow \quad \exists u \in A \; \exists n \geq 1 \text{ such that } g^n = hf.$$

Conclude the following facts:

(i) If $U = X_f$ for some $f \in A$, then the ring $A(U) := A_f$ depends only on U and not on f. Here A_f is the localization of A along f.

(ii) If $U' = X_g$ is another basic open set such that $X_g \subset X_f$, define a homomorphism $\rho = \rho_{U'U} : A(U) \longrightarrow A(U')$. We call ρ the restriction homomorphism.

(iii) The restriction homomorphisms satisfy $\rho_{UU} = id$ and for every inclusion $U'' \subset U' \subset U$ of basic open sets, we have $\rho_{U''U} = \rho_{U''U'} \circ \rho_{U'U}$.

(iv) Let $x = \mathfrak{p}$ be a point of X. Then

$$\varinjlim_{x \in U} A(U) \cong A_{\mathfrak{p}}$$

Here the direct limit is taken over the directed system $(\{A(U)\}_U, \{\rho_{U'U}\}_{U'U})$ where the index set runs over all basic open sets U that contain x and the ordering is given by inclusion.

The assignment of the ring $A(U)$ for every basic open set U and the restriction homomorphisms $\rho_{U'U}$ for every pair $U' \subset U$ of basic open sets, that satisfy (iii) above, defines a *presheaf of rings* on the basis of open sets $(X_f)_{f \in A}$ of X. (iv) shows that the stalk of this presheaf at $x \in X$ is the corresponding local ring $A_{\mathfrak{p}}$.

3. In the notation from the previous exercise, let $(U_i)_{i\in I}$ be a covering of X by basic open sets. Prove:
If $s_i \in A(U_i)\ \forall i \in I$ is given such that for every i, j the images of s_i and s_j in $A(U_i \cap U_j)$ are equal (i.e. such that $\rho_{U_i\cap U_j,U_i}(s_i) = \rho_{U_i\cap U_j,U_j}(s_j)\forall i, j$), then there exists a unique $s \in A = A(X)$ such that $\rho_{UX}(s) = s_i$ for all $i \in I$. Essentially, this implies that the presheaf is a *sheaf*.

4. Let A be a ring and let $\mathfrak{p}$ be a prime ideal of A. Show that the canonical image of $\mathrm{Spec}(A_\mathfrak{p})$ in $\mathrm{Spec}(A)$ in the intersection of all neighborhoods of $\mathfrak{p}$ in $\mathrm{Spec}(A)$.

5. Let A and B be two rings and let $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$ be their spectra.

(i) Let $S \subset A$ be a multiplicative subset and denote with $\phi : A \longrightarrow S^{-1}A$ the localization map. Show that $\phi^* : \mathrm{Spec}(S^{-1}A) \longrightarrow \mathrm{Spec}(A)$ is a homeomorphism onto its image in X. We will denote this image by $S^{-1}X$. For $f \in A$, show that the image of $\mathrm{Spec}(A_f)$ in $\mathrm{Spec}(A)$ is just the basic open subset X_f. As a result, we can identify $\mathrm{Spec}(S^{-1}A)$ with its image $S^{-1}X$ in X and we will do so below.

(ii) Let $f : A \longrightarrow B$ be a ring homomorphism and let $f^* : Y \longrightarrow X$ be the induced mapping. Let $S^{-1}Y = \mathrm{Spec}(S^{-1}B) := \mathrm{Spec}(f(S)^{-1}B)$ be the open subset of Y associated to the multiplicative set $f(S) \subset B$. Show that the induced mapping

$$(S^{-1}f)^* : \mathrm{Spec}(S^{-1}B) \longrightarrow \mathrm{Spec}(S^{-1}A)$$

is the restriction of $f^* : Y \longrightarrow X$ to $\mathrm{Spec}(S^{-1}B)$.

(iii) Let $\mathfrak{a}$ be an ideal of A and let $\phi : A \longrightarrow A/\mathfrak{a}$ be the canonical homomrphism. Show that $\phi^* : \mathrm{Spec}(A/\mathfrak{a}) \longrightarrow \mathrm{Spec}(A)$ gives an homeomorphism onto its image $V(\mathfrak{a})$ in X. Hence we can identify $\mathrm{Spec}(A/\mathfrak{a})$ with its image $V(\mathfrak{a})$ and we will do so below.

(iv) Again, let $\mathfrak{a}$ be an ideal in A and let $\mathfrak{b} = \mathfrak{a}^e$ be its extension in B. Denote with $\bar{f} : A/\mathfrak{a} \longrightarrow B/\mathfrak{b}$ the induced homomorphism. Show that the mapping

$$\bar{f}^* : \mathrm{Spec}(B/\mathfrak{b}) \longrightarrow \mathrm{Spec}(A/\mathfrak{a})$$

is the restriction of $f^* : Y \longrightarrow X$ to $V(\mathfrak{b})$.

(v) Let $\mathfrak{p}$ be a prime ideal in A. Show that the subspace $(f^*)^{-1}(\mathfrak{p})$ of Y is canonically homeomorphic to $\mathrm{Spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}) = \mathrm{Spec}(k(\mathfrak{p}) \otimes_A B)$, where $k(\mathfrak{p})$ is the residue field of the local ring $A_\mathfrak{p}$. (Hint: First use (ii) for the multiplicative subset $S = A \setminus \mathfrak{p}$ and then (iv) for the ideal $S^{-1}\mathfrak{p} \subset S^{-1}A$.) $\mathrm{Spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p})$ is called the *fiber* of f^* over $\mathfrak{p}$.

Due on Thursday, 05.12.2013