Exercise Sheet 11

Exercises **1. - 8.** are taken from the book *Introduction to Commutative Algebra* by Atiyah and Macdonald.

(i) Let X be a topological space. X is said to be *Noetherian* if the open sets in X satisfy the ascending chain condition, i.e. that every chain

$$U_1 \subset U_2 \subset U_3 \subset \dots$$

of increasing open subsets of X becomes stationary. Equivalently, the set of closed subsets satisfy the descending chain condition.

(ii) A nonempty subset Y of X is *irreducible* if it cannot be expressed as the union $Y = Y_1 \cup Y_2$ of two proper subsets, each one of which is closed in Y.

(iii) An *irreducible component* of X is a maximal irreducible subset of X.

1. Let X be a topological space. Show that, if X is Noetherian, then every subspace of X is Noetherian, and that X is quasi-compact.

2. Prove that the following conditions are equivalent.

(i) X is Noetherian

(ii) Every open subspace of X is quasi-compact

(iii) Every subspace of X is quasi-compact.

3. Show that a Noetherian space is a *finite* union of irreducible closed subspaces. (*Hint:* Let Σ be the set of closed subsets of X, which are not a finite union of irreducible closed subspaces. If we assume that Σ is non-empty, then there exists a minimal element.) We conclude that the set of irreducible components of a Noetherian space is finite.

4. Let A be a Noetherian ring. Show that, Spec(A) is a Noetherian topological space. Is the converse also true? (Hint: Consider $k[x_1, x_2, x_3, \dots]/(x_1, x_2^2, x_3^3, \dots)$)

5. Show that the set of minimal prime ideals in a Noetherian ring A is finite. (Use Exercise 3.)

6. Let X be a topological space and let $\mathcal{F}$ be the smallest collection of subsets of X which contains all open subsets of X and is closed under finite intersections and complements.

(a) Show that a subset E of X belongs to $\mathcal{F}$ if and only if E is a finite union of sets of the form $U \cap C$, where U is open and C is closed.

(b) Suppose that X is irreducible and let $E \in \mathcal{F}$. Show that E is dense in X (i.e. $\overline{E} = X$) if and only if E contains a non-empty open set in X

7. Let X be a Noetherian topological space and let $E \subset X$. Show that $E \in \mathcal{F}$ if and only if, for each irreducible closed set $X_0 \subset X$, either $\overline{E \cap X_0} \neq X_0$ or else $E \cap X_0$ contains a non-empty open subset of X_0. (*Hint*: Suppose $E \notin \mathcal{F}$. Then the collection of closed sets $X' \subset X$ such that $E \cap X' \notin \mathcal{F}$ is not-empty and has therefore a minimal element X_0. Show that X_0 is irreducible and that each of the alternatives above leads to the conclusion $E \cap X_0 \in \mathcal{F}$.) We call the sets in $\mathcal{F}$ the *constructible* subsets of X.

8. Let X be a Noetherian topological space and let E be a subset of X. Show that E is open in X if and only if, for each irreducible closed subset X_0 in X, either $E \cap X_0 = \varnothing$ or else $E \cap X_0$ contains a non-empty open subset of X_0. (Use a similar argument as in problem 7).

Due on Thursday, 12.12.2013