Commutative Algebra

**D-MATH, HS 2013**

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# Exercise Sheet 12

Exercises **1. - 4.** are taken from the book *Introduction to Commutative Algebra* by Atiyah and Macdonald.

Recall that a ring or module is Artin, if it satisfies the descending chain condition.

**1.** Let $M$ be an Artinian $A$-module. Suppose that $u : M \longrightarrow M$ is an injective homomorphism. Show that $u$ is an isomorphism. (*Hint*: Consider the quotient modules $\mathrm{Coker}(u^n)$. This is dual to Problem 2 of Sheet 8.)

**2.** Let $M$ be an Artinian $A$-module and let $\mathfrak{a}$ be the annihilator of $M$ in $A$. Is $A/\mathfrak{a}$ Artinian?

**3.** Let $A$ be a Noetherian ring. Show that the following are equivalent:

i) $A$ is Artinian.

ii) $\mathrm{Spec}(A)$ is discrete and finite.

iii) $\mathrm{Spec}(A)$ is discrete.

**4.** Let $k$ be a field and $A$ a finitely generated $k$-algebra. Show that the following are equivalent:

i) $A$ is Artinian.

ii) $A$ is a finite $k$-algebra.

(*Hint*: For i) $\rightarrow$ ii), use the fact that every Artinian ring is a finite product of local Artinian rings to reduce to the case of a Artin local ring. By the Nullstellensatz the residue field is a finite extension of $k$. Finally use that the length of $A$ as an $A$-module is finite. For ii) $\rightarrow$ i), use the descending chain condition for the ideals of $A$, which are $k$-vector spaces.)

**5.** Let $k$ be an algebraically closed field and let $A$ be a finitely generated $k$-algebra, that is Artin. The previous exercise showed that $A$ is finite-dimensional over $k$. What are the possible such $A$ of dimension 2 and 3?

**6.** Let $A = \mathbb{C}[x, y, z]$ be the polynomial ring with the standard degree grading. Consider the graded ideals

i) $J_1 = (x^3 + y^3 + z^3)$

ii) $J_2 = (x^3 + y^3 + z^3, x^2 + y^2 + z^2)$.

Calculate the series $P(J_i, t), P(\mathbb{C}[x, y, z]/J_i, t)$ for $i = 1, 2$.

**Due on Thursday, 19.12.2013**