Commutative Algebra

D-MATH, HS 2013

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Exercise Sheet 13

Exercises **3.** - **5.** are taken from the book *Introduction to Commutative Algebra* by Atiyah and Macdonald.

1. Let f be a homogeneous polynomial of degree d in the variables $x_0, \dots, x_n$. Calculate the Hilbert polynomial of $k[x_0, \dots, x_n]/f$. What is the leading term?

2. Let $\mathfrak{p}_r = (x_1, ..., x_r)$ in $\mathbb{C}[x_1, ..., x_s]$ for $r \leq s$. What is the Krull dimension of the localization $\mathbb{C}[x_1, ..., x_s]_{\mathfrak{p}_r}$?

3. Let A be a ring such that

(i) for each maximal ideal $\mathfrak{m}$ of A, the local ring $A_{\mathfrak{m}}$ is Noetherian;

(ii) for each $x \neq 0$ in A, the set of maximal ideals of A which contain x is finite.

Show that A is Noetherian

Hint: Let $\mathfrak{a} \neq 0$ be an ideal in A. Let $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ be the maximal ideals which contain $\mathfrak{a}$. Choose $x_0 \neq 0$ in $\mathfrak{a}$ and let $\mathfrak{m}_1, \dots, \mathfrak{m}_{r+s}$ be the maximal ideals which contain x_0. Since $\mathfrak{m}_{r+1}, \dots, \mathfrak{m}_{r+s}$ do not contain $\mathfrak{a}$ there exist $x_j \in \mathfrak{a}$ such that $x_j \notin \mathfrak{m}_{r+j}$ $(1 \leq j \leq s)$. Since each $A_{\mathfrak{m}_i}$ $(1 \leq i \leq r)$ is Noetherian, the extension of $\mathfrak{a}$ in $A_{\mathfrak{m}_i}$ is finitely generated. Hence there exist $x_{s+1}, \dots, x_t$ in $\mathfrak{a}$ whose images in $A_{\mathfrak{m}_i}$ generate $A_{\mathfrak{m}_i}\mathfrak{a}$ for $i = 1, \dots, r$. Let $\mathfrak{a}_0 = (x_0, \dots, x_t)$. Show that $\mathfrak{a}_0$ and $\mathfrak{a}$ have the same extension in $A_{\mathfrak{m}}$ for *every* maximal ideal $\mathfrak{m}$. By Prop 3.9 in AM, deduce that $\mathfrak{a}_0 = \mathfrak{a}$.

4. Following Nagata we will construct a Noetherian integral domain of infinite dimension. Let k be a field and let $A = k[x_1, x_2, \dots]$ be a polynomial ring over k in the countably many variables $x_i, i \in \mathbb{N}$. Let $m_1, m_2, \dots$ be an increasing sequence of positive integers such that $m_{i+1} - m_i > m_i - m_{i-1}$ for all $i > 1$. Let $\mathfrak{p}_i = (x_j \mid j = m_i + 1, \dots, m_{i+1})$ and let S be the complement in A of the union of the ideals $\mathfrak{p}_i$. As $\mathfrak{p}_i$ is prime, S is a multiplicatively closed set. Show:

(i) $S^{-1}A$ is Noetherian by Exercise 3 above.

(ii) Each $S^{-1}\mathfrak{p}_i$ has height $m_{i+1} - m_i$.

Conclude that $\dim S^{-1}A = \infty$.

5. Let A be a ring and let M be an A-module. The *support* of M is defined to be the set $\mathrm{Supp}(M)$ of prime ideals $\mathfrak{p}$ of A such that $M_{\mathfrak{p}} \neq 0$. Prove the following results:

(a) $M \neq 0 \iff \mathrm{Supp}(M) \neq \varnothing$.

(b) $V(\mathfrak{a}) = \mathrm{Supp}(A/\mathfrak{a})$.

(c) If $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is an exact sequence, then $\mathrm{Supp}(M) = \mathrm{Supp}(M') \cup \mathrm{Supp}(M'')$.

(d) If $M = \oplus M_i$, then $\mathrm{Supp}(M) = \cup\, \mathrm{Supp}(M_i)$.

(e) If M is finitely generated, then $\mathrm{Supp}(M) = V(\mathrm{Ann}(M))$. In particular, $\mathrm{Supp}(M)$ is a closed subset of $\mathrm{Spec}\, A$.

(f) If M, N are finitely generated, then $\mathrm{Supp}(M \otimes_A N) = \mathrm{Supp}(M) \cap \mathrm{Supp}(N)$. (Use problem 2 on sheet 4.)

(g) If M is finitely generated and $\mathfrak{a}$ is an ideal of A, then $\mathrm{Supp}(M/\mathfrak{a}M) = V(\mathfrak{a} + \mathrm{Ann}(M))$.

(h) If $f : A \longrightarrow B$ is a ring homomorphism and M is a finitely generated A-module, then $\mathrm{Supp}(B \otimes_A M) = f^{*-1}(\mathrm{Supp}(M))$.

6. Let $(A, \mathfrak{m})$ be a Noetherian local integral domain. Let $k = A/\mathfrak{m}$ be the residue field and let K be the quotient field of A. Let M be a finitely generated A-module. Show that if $\dim M \otimes_A k = \dim M \otimes_A K = r$, then M is free of rank r.

7. Let A be a ring. Let M be a finitely presented A-module and let N be a A-module. Show that

$$S^{-1} \mathrm{Hom}_A(M, N) = \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$$

for all multiplicative subsets S of A. Why does your proof fail if M is only finitely generated?

Happy Holidays!