Commutative Algebra

D-MATH, HS 2013

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Exercise Sheet 2

All exercises are taken from chapter 1 of the book *Introduction to Commutative Algebra* by Atiyah and Macdonald.

1. Let A be a ring such that $\forall x \in A\ \exists n > 1$ such that $x^n = x$. Show that every prime ideal is maximal.

2. Let A be a ring. For each $f \in A$ let X_f denote the complement of $V(f)$ in $X = \operatorname{Spec}(A)$. The sets X_f are called *basic open sets* of X. Show that

(a) the basic open sets form a basis of the Zariski topology.

(b) $X_f \cap X_g = X_{fg}$.

(c) $X_f = \varnothing$ if and only if f is nilpotent.

(d) $X_f = X$ if and only if f is a unit.

(e) $X_f = X_g$ if and only if $r((f)) = r((g))$.

(f) X is quasi-compact [1] and that an open subset of X is quasi-compact if and only if it is a finite union of basic open sets.

3. Let $X = \operatorname{Spec}(A)$ and $\mathfrak{p} \in X$ a prime ideal. Show

(a) $\{\mathfrak{p}\}$ is closed $\Longleftrightarrow$ $\mathfrak{p}$ is maximal.

(b) $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$ with $V(\mathfrak{p})$ defined in Exercise 6 of Sheet 1.

(c) X is a T_0 space (A space X is T_0 if for distinct points x, y in X, either there is a neighborhood of x which does not contain y, or a neighborhood of y that does not contain x.)

4. A topological space X is irreducible if $X \neq \varnothing$ and if every two non-empty open sets in X intersect. This is equivalent to the statement that every non-empty open set is dense in X. Show that $\operatorname{Spec}(A)$ is irreducible if and only if the nilradical of A is prime.

5. Let $\varphi : A \to B$ be a ring homomorphism. Let $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$. If $\mathfrak{q} \in Y$, then $\varphi^{-1}(\mathfrak{q})$ is an element of X, hence φ induces a map $\varphi^* : Y \to X$.

(a) If $f \in A$ then $(\varphi^*)^{-1}(X_f) = Y_{\varphi(f)}$. Conclude that φ^* is continuous.

(b) Show that $\operatorname{Spec}(-)$ induces a contravariant functor from the category of commutative rings to the category of topological spaces.

[1] A space X is quasi-compact if every open cover has a finite subcover.

(c) If $\mathfrak{a}$ is an ideal of A, then $(\varphi^*)^{-1}(V(\mathfrak{a})) = V(\mathfrak{a}^e)$.

(d) If $\mathfrak{b}$ is an ideal of B, then $\overline{\varphi^*(V(\mathfrak{b}))} = V(\mathfrak{b}^c)$.

(e) If φ is surjective, then φ^* is a homeomorphism of Y onto the closed subset $V(\mathrm{Ker}(\varphi))$ of X. (In particular, $\mathrm{Spec}(A)$ and $\mathrm{Spec}(A/\mathfrak{N})$ are naturally isomorphic, where $\mathfrak{N}$ is the nilradical of A.)

(f) If φ is injective, then $\varphi^*(Y)$ is dense in X. More precisely, $\varphi^*(Y)$ is dense in X if and only if $\mathrm{Ker}(\varphi) \subseteq \mathfrak{N}$.

(g) Let A be an integral domain with just one non-zero prime ideal $\mathfrak{p}$, and let K be the field of fractions of A. Let $B = (A/\mathfrak{p}) \times K$. Define $\varphi : A \to B$ by $\varphi(x) = (\overline{x}, x)$, where $\overline{x}$ is the image of x in $A/\mathfrak{p}$. Show that φ^* is bijective but not a homeomorphism.

6. Let k be an algebraically closed field. Let $\{f_\alpha(t_1, \ldots, t_n)\}_{\alpha \in A}$ be a set of polynomial equations in n variables with coefficients in k and consider the set X of all points $x = (x_1, \ldots, x_n) \in k^n$ which satisfy these equations. This is called an *affine algebraic variety.*

Let $I(X)$ be the set of all polynomials $g \in k[t_1, \ldots, t_n]$ such that $g(x) = 0\ \forall x \in X$. This is an ideal and called the *ideal associated to* X. The quotient ring $A(X) = k[t_1, \ldots, t_n]/I(X)$ is the ring of polynomial functions on X (two polynomials are the same polynomial function on X, if their difference vanishes on X) and we call it the *coordinate ring* of X.

Let ξ_i be the image of t_i in $A(X)$. We call $\xi_i, i = 1, \ldots, n$ the *coordinate functions* of X, because if $x \in X$, then $\xi_i(x)$ is the ith coordinate. Let's call $Max(A(X))$ the set of *maximal* ideals of $A(X)$. Define a mapping

$$\mu : X \longrightarrow Max(A(X)), \quad x \mapsto \mathfrak{m}_x$$

where $\mathfrak{m}_x = \{f \in A(X) \mid f(x) = 0\}$. Show that μ is injective. Indeed, we will later see that for an algebraically closed field this map gives a correspondence between points in X and maximal ideals in $A(X)$. This follows from Hilbert's Nullstellensatz.

Due on Tuesday, 10.10.2013