Commutative Algebra

D-MATH, HS 2013

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Exercise Sheet 3

Exercises **2.**, **3.** and **5.** are taken from chapter 2 of the book *Introduction to Commutative Algebra* by Atiyah and Macdonald.

1. Give a complete proof of the Snake Lemma.

2. Let $\mathfrak{p}$ be a prime ideal in a ring A. Show that $\mathfrak{p}[x]$ is a prime ideal in $A[x]$. Is the corresponding statement for maximal ideals also true?

3. Let M be a finitely generated A-Module and $\varphi : M \longrightarrow A^n$ a surjective homomorphism. Show that $\mathrm{Ker}(\varphi)$ is finitely generated.

4. Let $M \cong A^{\oplus n}$ be a free module of rank n. Use the Nakayama lemma to show, that every set of n generators is a basis.

Below we will introduce the notion of a *direct limit* of a *direct system* of A-Modules. We have the following definitions:

a) *directed set*: A partially ordered set I is said to be *directed* if for each pair i, j in I, there is a k such that $k \geq i$ and $k \geq j$.

b) *direct system of A-Modules*: Let A be a ring, I a directed set and $(M_i)_{i\in I}$ a family of A-modules, so that for each pair i, j with $i \leq j$ there exists a A-homomorphism $\mu_{ij} : M_i \longrightarrow M_j$ satisfying the following two conditions:

   i) μ_{ii} is the identity for all $i \in I$.

   ii) $\mu_{ik} = \mu_{jk} \circ \mu_{ij}$.

   Then we call $\mathbf{M} = (M_i, \mu_{ij})$ a *direct system* over the directed set I.

c) *direct limit of* $\mathbf{M}$: Let $C = \oplus_{i\in I} M_i$ be the direct sum of the M_i and identify each module M_i with its canonical image in C. Let D be the submodule of C generated by all elements of the form $x_i - \mu_{ij}(x_i)$ for $i \leq j$ and $x_i \in M_i$. Set $M = C/D$ and denote with $\mu_i : M_i \longrightarrow M$ the canonical map (that is, μ_i is defined as the inclusion $M_i \longrightarrow C$ followed by the projection $C \longrightarrow M$). Then we call the pair consisting of M and the family of homomorphism $\mu_i : M_i \longrightarrow M$ for $i \in I$ the *direct limit* of the direct system $\mathbf{M}$ and denote it by $\varinjlim_{i\in I} M_i$.

5. For the direct limit M of a directed system of A-Modules $\mathbf{M}$, show that:

   a) $\mu_i = \mu_j \circ \mu_{ij}$

b) for every $x \in M$, there is a $i \in I$ and a $x_i \in M_i$ such that $x = \mu_i(x_i)$

c) if $x_i \in M_i$ such that $\mu_i(x_i) = 0$, there exists a $j \geq i$ such that $\mu_{ij}(x_i) = 0$

d) the direct limit is characterized (up to isomorphism) by the following (universal) property: For any A-module N and for any collection of maps $\alpha_i : M_i \longrightarrow N$, $i \in I$ such that $\alpha_i = \alpha_j \circ \mu_{ij}$ whenever $j \geq i$, there exists a unique homomorphism $\alpha : M \longrightarrow N$ such that $\alpha_i = \alpha \circ \mu_i$ for all $i \in I$.

6. Let $\mathbf{M} = (M_i, \mu_{ij})$ be a directed system of A-Modules and let M be the direct limit. We want to give an alternative description of the direct limit of $\mathbf{M}$. Define the set

$$C = \Big(\bigsqcup_{i \in I} M_i\Big) \,/ \sim$$

where $(x_i, i) \sim (x_j, j)$ if and only if there is a $k \in I$ with $k \geq i$ and $k \geq j$ so that $\mu_{ik}(x_i) = \mu_{jk}(x_j)$. Define addition $+$ as follows: For $a = [(x_i, i)]$ and $b = [(x_j, j)]$ in C, let $k \in I$ so that $k \geq i, j$ and set $a + b = [(\mu_{ik}(x_i) + \mu_{jk}(x_j), k)]$.

a) Show that $\sim$ is an equivalence relation.

b) Prove that the addition operation $+$ is well defined.

c) Define a multiplication operation $\cdot : A \times C \longrightarrow C$ and show that it is well defined.

d) Prove that $(C, +, \cdot)$ is a A-Module.

e) Finally show that C is isomorphic to the direct limit M.

Due on Tuesday, 17.10.2013