Exercise Sheet 4

1. Prove the following equalities:

(a) $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/\gcd(m,n)\mathbb{Z}$,

(b) $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$,

(c) $(\mathbb{Q}/\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z}) = 0$,

(d) For a ring A, an ideal $\mathfrak{a}$ of A and an A-module M we have $(A/\mathfrak{a}) \otimes_A M \cong (M/\mathfrak{a}M)$.

(e) Let $A \subseteq B$ be a ring extension. Then $B \otimes_A A[X] \cong B[X]$.

(f) $A[X] \otimes_A A[Y] \cong A[X,Y]$.

2. Let A be a local ring, M and N finitely generated A-Modules. Prove that if $M \otimes N = 0$, then $M = 0$ or $N = 0$. *Hint:* Use Nakayama and part d) from the previous exercise.

Definition. A *Category* $\mathcal{C}$ consists of

i) a collection of objects $Ob(\mathcal{C})$

ii) for any two objects $A, B \in Ob(\mathcal{C})$ a set $Mor(A,B)$ called the *morphisms* from A to B

iii) for for any three objects $A, B, C \in Ob(\mathcal{C})$ a composition map

$$\circ : Mor(B,C) \times Mor(A,B) \longrightarrow Mor(A,C), (f,g) \mapsto f \circ g$$

such that the following axioms are true:

a) $Mor(A,B)$ and $Mor(A',B')$ are disjoint unless $A = A'$ and $B = B'$.

b) For objects A, B, C, D and morphisms $f \in Mor(A,B), g \in Mor(B,C)$, $h \in Mor(C,D)$ we have

$$(h \circ g) \circ f = h \circ (g \circ f) \qquad \text{(Associativity)}$$

c) For any object $A \in Ob(\mathcal{C})$ there is a morphism $id_A \in Mor(A,A)$ such that for any $f \in Mor(A,B)$ and any $g \in Mor(C,A)$ we have

$$f \circ id_A = f, \qquad id_A \circ g = g \qquad \text{(Identity element)}$$

Common examples of categories are the category of sets, denoted *Sets*, whose objects are sets and morphisms are maps between sets, the category *Top*, whose objects are topological spaces and morphisms are continuous maps between them, or the category Mod_A of A-modules over a ring A, with objects A-modules and morphisms A-homomorphisms.

3. Let $\mathcal{C}$ be a category and A, B two objects of $\mathcal{C}$. We say that an object $P \in Ob(\mathcal{C})$ together with two maps $a : P \longrightarrow A$, $b : P \longrightarrow B$ is a product of A and B when it satisfies the following universal property:
For any object C and every two maps $f : C \longrightarrow A$, $g : C \longrightarrow B$ there exists a unique map $h : C \longrightarrow P$ such that $a \circ h = f$, $b \circ h = g$.

i) Prove that if a product exists, it is unique up to a canonical isomorphism.

We will denote the product of A and B by $A \times B$.

ii) Show that the usual product of two sets satisfies the universal property of a product in the category of sets *Sets*.

iii) Convince yourself that similar statements are true for *Top* and Mod_A.

iv) Suppose the product exists in a category $\mathcal{C}$. Show using the universal property that for any three objects A, B, C there is a canonical isomorphism

$$A \times (B \times C) \cong (A \times B) \times C$$

4. Let $\mathcal{C}$ be a category. Let A, B and Z be three objects of $\mathcal{C}$ and $f : A \to Z$, $g : B \to Z$ two morphisms. We say an object F together with maps $a : F \to A$, $b : F \to B$ such that $f \circ a = g \circ b$ is a *fibered product* of A and B over Z if it satisfies the following universal property:
For any object G and any two morphisms $a' : G \to A$, $b' : G \to B$ such that $f \circ a' = g \circ b'$, there is a unique morphism $k : G \to F$ such that $a' = a \circ k$ and $b' = b \circ k$.

i) If a fibered product exists, show that it is unique up to a canonical isomorphism. Hence we can call F the fibered product and denote it by $A \times_Z B$.

ii) Show that the fibered product exists in the category of Sets, in particular show that for two sets A, B we have

$$A \times_Z B = \{(x, y) \in A \times B \mid f(x) = g(y)\}$$

In particular, if Z is a set with a single element, the fibered product reduces to the ordinary product.

Due on Tuesday, 24.10.2013