Commutative Algebra

D-MATH, HS 2013

Prof. Pandharipande

Exercise Sheet 5

Exercises **2. - 5.** are taken from the book *Introduction to Commutative Algebra* by Atiyah and Macdonald.

1. Let $\mathbb{Z}/a\mathbb{Z}$ be the ring of integers modulo a for some $a > 1$. Let $S = \{1, b, b^2, b^3, \dots\}$ be the multiplicative set associated to a non-zero integer b. Calculate the module $S^{-1}\mathbb{Z}/a\mathbb{Z}$.

2. Let S be a multiplicative subset of a ring A and let M be a finitely generated A-module. Show that $S^{-1}M = 0$ if and only if $\exists\ s \in S$ such that $sM = 0$.

Recall the two following definitions for a ring B:

· The *Jacobson ideal* of B is the intersection of all its maximal ideals.

· B is called *reduced* if there are no nilpotent elements in B except 0.

3. Let $\mathfrak{a}$ be an ideal of a ring A and let $S = 1 + \mathfrak{a}$. Show that $S^{-1}\mathfrak{a}$ is contained in the Jacobson radical of $S^{-1}A$. Use this result to prove Corollary 2.5 of Atiyah and Macdonald's book assuming Nakayama's Lemma (Proposition 2.6) in the following way: Suppose $\mathfrak{a}M = M$, then $S^{-1}M = (S^{-1}\mathfrak{a})(S^{-1}M)$ and therefore by the first part and Nakayama's Lemma, $S^{-1}M = 0$. By Exercise 2 this implies $s \cdot M = 0$ for some $s \in S$. Make this argument precise.

4. Let A be a ring. Show: If the localized ring $A_\mathfrak{p}$ is *reduced* for every prime ideal $\mathfrak{p}$ of A , then so is A. Does it analogously hold, that if $A_\mathfrak{p}$ is an integral domain for every prime ideal $\mathfrak{p}$, then so is A?

5. Let $\mathbf{M} = (M_i, \mu_{ij})$ be a directed system over a ring a and let $M = \varinjlim_{i\in I} M_i$ be its direct limit (see exercise sheet 3). Let N be an A-module. Then $(M_i \otimes N, \mu_{ij} \otimes 1)$ is a direct system and let $P = \varinjlim_{i\in I}(M_i \otimes N)$ be its direct limit. On the other hand, for each i, there is a homomorphism $\alpha_i = \mu_i \otimes 1 : M_i \otimes N \longrightarrow M \otimes N$ such that $\alpha_i = \alpha_j \circ \mu_{ij}$, hence by the universal property of the direct limit there is a homomorphism $\alpha : P \longrightarrow M \otimes N$. Show that

$$\alpha : \varinjlim_{i\in I} \big(M_i \otimes N\big) \longrightarrow \big(\varinjlim_{i\in I} M_i\big) \otimes N$$

is an isomorphism and hence that tensor product and direct limit commutes. (*Hint:* Construct an inverse to α as follows. Let $g_i : M_i \times N \longrightarrow M_i \otimes N$ be the universal pairing. Use the g_i to construct a A-bilinear map $M \times N \longrightarrow P$. By the universal property of P, this gives a map $\beta : M \otimes N \longrightarrow P$. Show that α and β are inverse to each other.)

Due on Tuesday, 31.10.2013