Commutative Algebra
D-MATH, HS 2013

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Exercise Sheet 7

Exercises **1. - 7.** are taken from the book *Introduction to Commutative Algebra* by Atiyah and Macdonald.

Recall: Let $f : A \longrightarrow B$ be a ring homomorphism and let $\mathfrak{a} \subset A$, $\mathfrak{b} \subset B$ be two ideals. Then the extension of $\mathfrak{a}$ and the contraction of $\mathfrak{b}$ is defined by $\mathfrak{a}^e = Bf(\mathfrak{a})$ and $\mathfrak{b}^c = f^{-1}(\mathfrak{b})$ respectively.

1. Let $f : A \longrightarrow B$ a ring homomorphism that makes B into a flat A-algebra. Show that the following conditions are equivalent:

(i) $(\mathfrak{a}^e)^c = \mathfrak{a}$ for all ideals $\mathfrak{a}$ of A.

(ii) $\mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$ is surjective.

(iii) For every maximal ideal $\mathfrak{m}$ of A, we have $\mathfrak{m}^e \neq (1)$.

(iv) If M is any non-zero A-module, then $M_B \neq 0$ (where $M_B = B \otimes_A M$).

(v) For every A-module M, the mapping $M \longrightarrow M_B, x \mapsto 1 \otimes x$ is injective.

If these conditions hold, we say B is *faithfully flat* over A.
(Hint: (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) is clear. (iii) $\Rightarrow$ (iv): let $x \in M$ and consider $M' = Ax$. By flatness, $M'_B \neq 0$ implies $M \neq 0$. Let $\mathfrak{a} = \mathrm{Ker}(A \longrightarrow M')$, choose a maximal ideal $\mathfrak{a} \subset \mathfrak{m}$ and apply (iii). (iv) $\Rightarrow$ (v): Apply $\otimes B$ to the exact sequence $0 \longrightarrow K \longrightarrow M \longrightarrow M_B$ and show that $M_B \longrightarrow (M_B)_B$ is injective. (v) $\Rightarrow$ (i): Use $M = A/\mathfrak{a}$.)

2. Let $f : A \longrightarrow B$ be a flat ring homomorphism and let $\mathfrak{q}$ be a prime ideal of B. Let $\mathfrak{p} = \mathfrak{q}^c$ be its contraction. Show that $\mathrm{Spec}(B_\mathfrak{q}) \longrightarrow \mathrm{Spec}(A_\mathfrak{p})$ is surjective. (Hint: Consider the composition $A_\mathfrak{p} \longrightarrow B_\mathfrak{p} \longrightarrow B_\mathfrak{q}$. Each individual map is flat, hence also the composition. Then use (iii) from problem 1.)

3. Let $A \xrightarrow{f} B \xrightarrow{g} C$ be ring homomorphisms such that $g \circ f$ is flat and g is faithfully flat. Prove that f is flat.

4. We show here another characterization of a valuation ring.

Definition. Let A and B be two local rings with $A \subset B$. B *dominates* A if the maximal ideal $\mathfrak{m}$ of A is contained in the maximal ideal $\mathfrak{n}$ of B, or equivalently $\mathfrak{m} = \mathfrak{n} \cap A$.

Let now K be a field and let Σ be the set of all local subrings of K. Define an ordering on Σ by the relation of domination and show that a ring $A \in \Sigma$ is maximal if and only if A is a valuation ring of K.

5. Let A be an integral domain and K its field of fractions. Show, that the following are equivalent:

(i) A is a valuation ring of K.

(ii) If $\mathfrak{a}, \mathfrak{b}$ are *any* two ideals of A, then either $\mathfrak{a} \subset \mathfrak{b}$ or $\mathfrak{b} \subset \mathfrak{a}$.

Using this equivalence show that if A is a valuation ring and $\mathfrak{p}$ a prime ideal of A, then $A_\mathfrak{p}$ and $A/\mathfrak{p}$ are valuation rings of their fields of fractions.

6. Let A be a valuation ring of a field K. Show that every subring B of K that contains A is a local ring of A, i.e. there is a prime ideal $\mathfrak{p}$ of A such that $B = A_\mathfrak{p}$.

7. Let A be a subring of a ring B with B integral over A. Let $\mathfrak{n}$ be a maximal ideal in B and let $\mathfrak{m} = \mathfrak{n} \cap A$ be the induced maximal ideal in A. Is $B_\mathfrak{n}$ integral over $A_\mathfrak{m}$? (*Hint*: Consider $A = k[x^2 - 1]$, $B = k[x]$ and the maximal ideal $\mathfrak{n} = (x - 1)$. Is $1/(x + 1)$ integral over $A_\mathfrak{m}$?)

Due on Tuesday, 14.11.2013