Commutative Algebra

D-MATH, HS 2013

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Exercise Sheet 8

Exercises **1. - 6.** are taken from the book *Introduction to Commutative Algebra* by Atiyah and Macdonald.

1. Which of the following rings are Noetherian?

(i) The ring of rational functions of z having no pole on the circle $|z| = 1$.

(ii) The ring of power series in z with a positive radius of convergence.

(iii) The ring of power series in z with an infinite radius of convergence.

(iv) The ring of polynomials in z whose first k derivatives vanish at the origin, where k is a fixed non-negative integer.

(v) The ring of polynomials in z, w whose partial derivatives with respect to w vanish for $z = 0$.

In all cases, the coefficients are complex numbers.

2. Let M be a Noetherian A-module and $u : M \longrightarrow M$ a module homomorphism. Show that, if u is surjective, then u is an isomorphism. (*Hint*: Consider $\mathrm{Ker}(u^n)$ for $n \geq 1$.)

3. Let $A[[x]]$ be the ring of power series in a variable x. Show that A Noetherian implies that so is $A[[x]]$.

4. The Hilbert basis theorem states that A Noetherian implies $A[x]$ Noetherian. Is the converse true, that if $A[x]$ is Noetherian, then also A must be Noetherian?

5. Let A be a Noetherian ring and let $f = \sum_{k=0}^{\infty} a_k x^k \in A[[x]]$. Prove that f is nilpotent if and only if each a_k is nilpotent.

6. Let A be a non-Noetherian ring and let Σ be the set of ideals in A which are not finitely generated. Show that Σ has maximal elements and that the maximal elements of Σ are prime. (*Hint*: Let $\mathfrak{a}$ be a maximal element of Σ and suppose there are $x, y \in A$ such that $x \notin \mathfrak{a}$ and $y \notin \mathfrak{a}$, but $xy \in \mathfrak{a}$. Show that there exist a finitely generated ideal $\mathfrak{a}_0 \subset \mathfrak{a}$ such that $\mathfrak{a}_0 + (x) = \mathfrak{a} + (x)$ and $\mathfrak{a} = \mathfrak{a}_0 + x \cdot (\mathfrak{a} : x)$. Since $\mathfrak{a} \subsetneq (\mathfrak{a} : x)$, it follows that $(\mathfrak{a} : x)$ is finitely generated and so $\mathfrak{a}$. Contradiction.)

We have shown that a ring in which every *prime* ideal is finitely generated is Noetherian.

Due on Tuesday, 21.11.2013