Exercise Sheet 9

Exercises **1. - 7.** are taken from the book *Introduction to Commutative Algebra* by Atiyah and Macdonald.

1. Let A be a Noetherian ring, B a finitely generated A-algebra and G a finite group of A-automorphisms of B. Let

$$B^G = \{b \in B \mid g \cdot b = b \text{ for all } g \in G\}$$

be the set of G-left-invariant elements of B. Show that B^G is a finitely generated A-algebra.

2. Let A be a ring such that each local ring $A_{\mathfrak{p}}$ is Noetherian. Does this imply A Noetherian?

3. Let A be a ring and let B be a faithfully flat A-algebra (compare sheet 7). Show that B Noetherian implies A Noetherian (Hint: ascending chain condition).

4. Let M be a Noetherian A-module and let $\mathfrak{a}$ be the annihilator of M in A. Show that, $A/\mathfrak{a}$ is a Noetherian ring.

5. Let k be a field and B a finitely generated k-algebra. Suppose that B is a field. Then B is a finite algebraic extension of k.

Prove this in the following way. Let $x_1, \ldots, x_n$ generate B. Use induction on n. $n = 1$ is immediate, so assume $n > 1$ and that the statement holds for $n - 1$ generators. Let $A = k[x_1]$ and let $K = k(x_1)$ be the field of fractions of A. By induction, B is a finite algebraic extension of K, hence each $x_2, \ldots, x_n$ satisfies a monic polynomial equation with coefficients of K. The coefficients are of the form a/b with $a, b \in A$. If f is the product of the denominators of all these coefficients, then each $x_2, \ldots, x_n$ is integral over A_f. This implies that B and therefore K is integral over A_f.

Suppose that x_1 is transcendental over k. Then A is integrally closed, because it is a unique factorization domain. Hence A_f is integrally closed, and therefore $A_f = K$, which is a contradiction. Hence x_1 is algebraic over k and K a finite extension of k. This implies that B is a finite extension of k.

6. Let A be a valuation ring of a field K and denote with U the set of units of A. U is naturally a subgroup of the multiplicative group K^* of the field K.

Let $\Gamma = K^*/U$ be the quotient group. If $\xi, \eta \in \Gamma$ are represented by $x, y \in K$, define

$$\xi \geq \eta \qquad \Longleftrightarrow \qquad xy^{-1} \in A$$

Show:

(i) $\geq$ defines a total ordering on Γ.

(ii) $\geq$ is compatible with the group structure on Γ, i.e. show that if $\xi \geq \eta$ and $\omega \in \Gamma$, then $\xi\omega \geq \eta\omega$.

(iii) Let $\nu : K^* \longrightarrow \Gamma$ be the projection. Show that for all $x, y \in K^*$ we have $\nu(x+y) \geq \min(\nu(x), \nu(y))$.

Properties (i) and (ii) show that Γ is a totally ordered abelian group. It is called the *value group* of A. The map ν is called a *valuation* of the field K with value group Γ.

7. Show conversely that every valuation of a field K is induced by a valuation ring. More precisely, let Γ be a totally ordered abelian group, with group operation $+$ and let K be a field. A *valuation* of K with values in Γ is a map $\nu : K^* \longrightarrow \Gamma$ such that for all $x, y \in K^*$

(i) $\nu(xy) = \nu(x) + \nu(y)$

(ii) $\nu(x+y) \geq \min(\nu(x), \nu(y))$.

Show that the set of elements $x \in K^*$ such that $\nu(x) \geq 0$ is a valuation ring of K. This ring is called the *valuation ring* of ν and the subgroup $\nu(K^*)$ of Γ is the *value group* of ν.

Due on Thursday, 28.11.2013