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# INVERSE PROBLEMS

## Part I : Linear inverse problems

### 1.1 Examples

#### 1.1.1: Preface Ill-posed linear operator equations

$X, Y \hat{=}$ Banach spaces, even Hilbert spaces

$T \in \mathcal{L}(X,Y)$ bounded linear operator called forward operator, its evaluation is the forward problem / direct problem.

Operator equation: seek $f \in X$ :

$$Tf = g \qquad (1.1.1A)$$

for given $g \in Y$ (the data)

Often (1.1.1.A) will be ill-posed (Hadamard)

= not well posed

→
- existence of solution for all data
- uniqueness of solution
- the solution depends continuous on data ($\|g-\tilde{g}\|_Y$ small $\Rightarrow \|f-\tilde{f}\|_X$ small)

If (1.1.1A) is well-posed, then
- $T$ surjective
- $T$ injective
- $T^{-1}$ is bounded (because continuous + linear)

(first two ⟹ third, if $T$ linear and bounded between Banach spaces by open mapping theorem*)

* surjective (injective) bounded linear ops between Banach spaces are open

$\Rightarrow$ For ill-posed operator equations $T$ will fail to be surjective and fail to possess a bounded inverse.

Remark: Injective linear (not necessarily continuous) mapping $T: X \to Y$, introduce a norm on $R(T) = \{y \in Y : \exists x \in X : y = Tx\}$ range (linear space)

according to $\|y\|_* = \|x\|_X$ where $y = Tx$

$\Rightarrow \check{Y} = (R(T), \|\cdot\|_*)$ is a Banach space

(completeness: Cauchy sequence in $Y$ $\{y_n\}$

$\Rightarrow$ preimage of Cauchy sequence in $X$ (Banach space)
$\{x_n\}$

$\Rightarrow y \hat{=} \lim_n T(x_n)$)

$T\, X \to \check{Y}$ is surjective, $T^{-1}$ is bounded:

$$\|T^{-1}\hat{y}\|_X \underset{\text{def. of } \|\cdot\|_* \text{ norm}}{\hat{=}} \|T(T^{-1}\hat{y})\|_* = \|\hat{y}\|_*$$

$\Rightarrow \|T^{-1}\| = 1$.

$\Rightarrow$ the spaces (together with the norms) are really crucial to the problem to be ill-posed or not.

Ill-posed = posed on "wrong" spaces

but: (i) $\|\ \|_*$ may be awkward (not computable, not accessible)

(ii) norms $\|\ \|_X$, $\|\cdot\|_Y$ imposed by application context

(iii) perturbations of data $y$ can be estimated (controlled) in particular norms only

### 1.1.2 Differentiation

Data: $g: [0,1] \to \mathbb{R}$ 1-periodic

Sought: $f = g'$ $\quad (*)$

Associated operator

$$(Tf)(x) = \int_0^x f(\tau)\,d\tau \qquad (1.1.2A)$$

$(*) \Leftrightarrow$ operator equation $Tf = g$

$[\ (Tf)' = f \Rightarrow T$ injective $]$

~~Function space framework~~

~~(i) $X = C^0_{\text{periodic}}([0,1])$, $\int_0^1 f\,dx = 0$ (sup norm)~~

~~$Y = C^1_{\text{p}}([0,1])$ (sup norm + sup norm derivative)~~ ~~$T \in \mathcal{L}(X,Y)$~~

~~$T$ not surjective~~

(i) $X = \{ f \in C^0_{per}([0,1]),\ \int_0^1 f\,dx = 0 \}$

$Y = \{ g \in C^1_{per}([0,1]),\ g(0) = 0 \}$

$\Rightarrow T \in \mathcal{L}(X,Y)$, bijective

$\Rightarrow$ (1.1.2 A) well posed

(ii) ~~If the~~ However, perturbations in $g$ can be controlled in the maximum norm only

$$Y = \{ g \in C^0_{per}([0,1]),\ g(0) = 0 \}$$

Then $T$ may fail to be surjective
$\Rightarrow$ ill-posed problem.

Impact of perturbations in (this) $Y$
switch to $g \to g^\delta$ $\quad g^\delta(x) = g(x) + \delta \sin(n\pi x)$ $\quad \delta > 0,\ n \in \mathbb{N}$

$g^\delta$: noisy data

$\|g - g^\delta\|_Y \le \delta \ll 1$

$\delta$: <u>noise level</u>

Solution of $Tf^\delta = g^\delta$

$$f^\delta = g'(x) + \delta n\pi \cos(n\pi x)$$

$\Rightarrow \|f - f^\delta\|_X = n\delta\pi$

exact solution $f = g'$

Chose $n \approx \delta^{-2}$ (very large ~~s~~ if $\delta \ll 1$) $\Rightarrow \|f - f^\delta\| \approx \delta^{-1} \to \infty$ for $\delta \to 0$

$\Rightarrow$ no continuous dependence of solution from the data
$\Rightarrow$ ill-posed operator equation.

1.1.2 <u>Numerical differentiation</u>

Apply central difference quotient, $h > 0$

$$(R_h g)(x) := \frac{g(x+h) - g(x-h)}{2h} \quad (1.1.2.B)$$

In setting (ii) $R_h \in \mathcal{L}(Y, X)$, for all $h > 0$.
($h \,\hat{=}$ "discretization parameter)
$R_h \,\hat{=}$ reconstruction operator

$\Rightarrow$ reconstructed solution of $Tf = g$, $\hat{f}_\alpha = R_\alpha g$

Inevitable reconstruction error

$$R_\alpha T f - f = R_\alpha g - g' \quad \text{for } g \in R(T)$$

Estimate of reconstruction error:

Smoothness assumption $g \in C^2([0,1])$

$$g(x \pm h) = g(x) \pm g'(x) h + \frac{h^2}{2} g''(\xi) \quad \xi \in [x-h, x+h]$$

$\overset{\text{plug into } R_\alpha}{\Longrightarrow}$ $R_\alpha g(x) - g'(x) = \frac{1}{4} h \left( g''(\xi_1) + g''(\xi_2) \right)$

$\Rightarrow \| R_\alpha g - g'(x) \|_X \le \frac{1}{2} h \| g'' \|_\infty$ bounded by hp. on $g$

Total error (with perturbations)

$$\underset{\substack{\uparrow \\ \text{noisy data}}}{R_\alpha g^\delta} - g' \underset{\substack{\uparrow \\ R_\alpha \text{ linear}}}{=} \underbrace{R_\alpha(g^\delta - g)}_{\substack{\text{data} \\ \text{noise} \\ \text{error}}} + \underbrace{(R_\alpha g - g')}_{\substack{\text{reconstruction} \\ \text{error}}}$$