1.2 Differentiation

20-9-2013

$$(Tf)(x) = \int_0^x f(\tau)\, d\tau$$

$$\frac{d}{dx} Tf = f \quad \rightarrow T \text{ injective}$$

Consider $T: X \rightarrow Y$ on space

$$X = \left\{ v \in C^0_{per}([0,1]) : \int_0^1 v(x)\,dx = 0 \right\}$$

$$Y = C^0_{per}([0,1])$$

$\Rightarrow$ $Tf = g$ becomes ill-posed (i.e. T^{-1} is not bounded)

Noisy data: instead of $g \in R(T)$, only perturbed data $g^\delta \in Y$ are available.

Assumption: $\|g - g^\delta\|_Y \leq \delta$ ← noise level

Recovery of an approximation of f through "numerical differentiation"

$$R_h g^\delta(x) = \frac{g^\delta(x+h) - g^\delta(x-h)}{2h}$$

↑ a family of reconstruction operators

If $\|g''\|_\infty$ bounded, (→ a-priori information) then the total error can be bounded by

$$R_h g^\delta - g' = \underbrace{(R_h g - g')}_{\text{approximation error}} + \underbrace{R_h(g^\delta - g)}_{\text{data noise error}}$$

$\cancel{\|R_h g^\delta - g'\|}$

$$\|R_h g^\delta - g'\|_X \leq \frac{1}{2} h \|g''\|_\infty + \frac{1}{h}\|g^\delta - g\|_Y$$

$\lceil$ $R_h \varphi(x) \leq \frac{1}{h} \max_{x \in [0,1]} |\varphi(x)|$ for every $\varphi \in C^0_{per}$ ← important in this estimate because it guarantees that we can ~~…~~ computed φ for all x and all.

↑ triangle inequality

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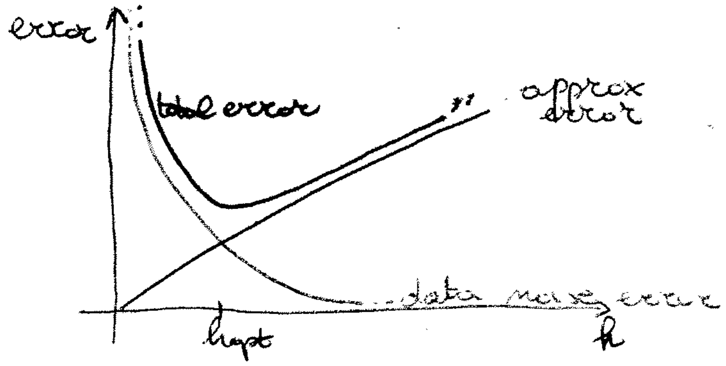
▷ Total error $\|R_a g^\delta - g'\|_X \le \frac{1}{2} h \|g''\|_\infty + \frac{1}{h}\delta$



Remark 1) total error bounded from below → bad
2) there is an optimal total error:

$$h_{opt} = \sqrt{\frac{2}{\|g''\|_\infty}}\sqrt{\delta}$$

▷ Minimal total error norm

$$h = h_{opt} \quad \|R_a^\delta - g\|_X = O(\delta^{\frac{1}{2}})$$

Remark If T^{-1} was bounded $\|T^{-1}(g-g^\delta)\| = O(\delta)$ (*)

we could achieve a recontruction of the same order of the perturbation ($O(\delta)$).
But here we achieve less: $O(\delta^{\frac{1}{2}})$

$\Longrightarrow$ (*) is worse, reflecting the ill-posed nature of the problem.

Remark (*) remarkable, because (although the problem was originally well-posed) small noise seems to allow good reconstruction, but only under the assumption of extra regularity of f: $\|f'\|_\infty$ bounded
this does not contradict the ill-posed nature of the problem

If $\|g'''\|_\infty$ bounded, use $\|R_a g - g'\|_\infty = \frac{1}{6} h^2 \|g'''\|_\infty$

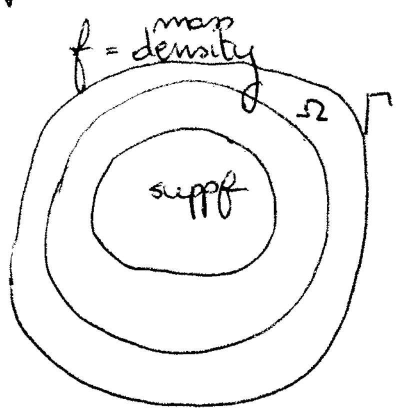
$\Longrightarrow \|R_a g^\delta - g'\|_X \le \frac{1}{6} h^2 \|g'''\|_\infty + \frac{1}{h} g$

$\Longrightarrow h_{opt} = \sqrt[3]{\frac{3}{\|g'''\|_\infty}} \sqrt[3]{\delta}$: $\|R_a g - g'\|_\infty = O(h^{\frac{2}{3}})$
h_{opt} will always depend on the noise level

The more you contrain the solution, the closer you get to the optimum (closer but never = because the pb is ill-posed)
$\to 0$ as $h \to 0$ not linearly?

1.13 Gravimetry

General: Inverse source problem for $-\Delta$



On Γ we measure gravity potential

$$-\Delta u = f$$

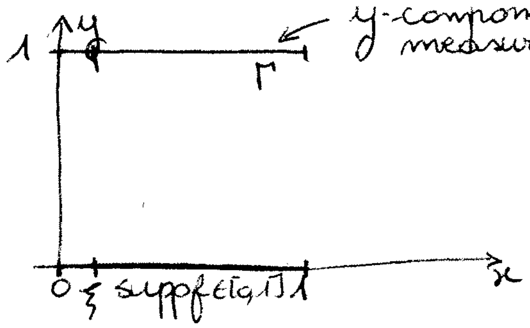
+ decay conditions

Find f, $\operatorname{supp} f \subset \Omega$ from measured values of u, grad u on Γ.

$\longrightarrow$ highly non-unique

When imposing assumptions on f, the problem becomes tractable.

1D version:



y-component of gravitational force measured here.

$f: [0,1] \longrightarrow \mathbb{R}_0^+ \mathrel{\hat{=}}$ mass density distribution on $[0,1]$.
(masses arranged on a line! : 3D!)

<u>Forward operator</u>

Force (y-component) generated by mass $f(x)\Delta x$

$$\delta F_y(\xi) = f(x)\delta x \frac{1}{\sqrt{(x-\xi)^2+1}^3}$$

$$\left(F(r) \sim \frac{r-r'}{\|r-r'\|^3} \right)$$

$\triangleright$ Total y-force $F_y(\xi) = \int_0^1 f(x) \frac{1}{\sqrt{(x-\xi)^2+1}^3}\, dx$

$$(Tf)(\xi) = \int_0^1 k(\xi - x)\, f(x)\, dx$$

with $k(\tau) = \dfrac{1}{\sqrt{\xi^2+1}^3}$

$T \mathrel{\hat{=}}$ convolution operator with kernel k.

Recall: convolution, $g, f : \mathbb{R} \to \mathbb{R}$

$$(g * f)(x) = \int_{\mathbb{R}} g(x-y)\, f(y)\, dy$$

Write $\tilde{f}$ for extension by zero of f outside $[0,1]$:

$$Tf = k * \tilde{f}$$

Result on convolution:

$$\begin{matrix} g \in L^p(\mathbb{R}) \\ f \in L^q(\mathbb{R}) \end{matrix}, \quad \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}, \quad 1 \le r \le \infty$$

$\infty > p, q \geq 1$

$$\Rightarrow \quad g * f \in L^r(\mathbb{R}), \quad \|g*f\|_{L^r} \le C \|g\|_{L^p} \|f\|_{L^q} \quad (*)$$
where $C = C(p,q)$

Apply this for $p = 1, q = 2 \Rightarrow r = 2$
because $k \in L^1(\mathbb{R})$ $\left[\int_{-\infty}^{+\infty} \frac{1}{\sqrt{\tau^2+1}^{\,3}} d\tau < \infty \right]$

$$\Rightarrow T : \underbrace{L^2(]0,1[)}_{= X} \longrightarrow \underbrace{L^2(]0,1[)}_{= Y} \text{ is continuous}$$

because (*) $\|Tf\|_{L^2} \le C \|k\|_{L^1} \underbrace{\|\tilde{f}\|_{L^2}}_{\overset{!}{=} \|f\|_{L^2}}$

Now let us study the inverse

Tool: Fourier transform

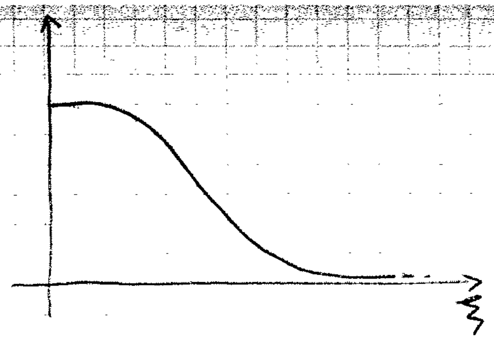
$$(\mathcal{F}\varphi)(\xi) = \int_{\mathbb{R}} \varphi(x)\, e^{-2\pi i x \xi}\, dx$$

Isometry: $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$
$\mathcal{F}$ "diagonalizes convolution":

$$\mathcal{F}(g * f) = \mathcal{F}(g)\, \mathcal{F}(f)$$

$L^1 \subset L^1$ for $g \in L^1(\mathbb{R}), f \in L^2(\mathbb{R})$

$\triangleright$ $\mathcal{F}(k)(\xi) = 4\pi |\xi| K_1(2\pi|\xi|)$
K_1: 2-nd Kind Bessel function



$F(k) > 0$

Formal inverse $\tilde{f} = \mathcal{F}^{-1}\left(\frac{\mathcal{F}(g)}{\mathcal{F}(k)}\right)^{(*)}$ for $Tf = g$

→ T is injective ($T^{-1}(0) = 0$)

→ Since $(\mathcal{F}(k)|\xi|) \to 0$ is $|\xi| \to \infty$,
$Tf = g$ is ill-posed:
choose ~~$g = \mathcal{F}^{-1}(\chi_{[n,n+1]})$ characteristic function~~ $g \neq 0$, $g \in R(T)$

$\mathcal{F}(g)$ supported in $[n, n+1]$ for $n \gg 1$

$\Longrightarrow \underbrace{\left\| \frac{\mathcal{F}(g)}{\mathcal{F}(k)} \right\|_{L^2}}_{\|f\|_{L^2}} \gg \|\mathcal{F}(g)\|_{L^2} = \|g\|_{L^2}$ for large n (because $\mathcal{F}(k)$ is very small there).

Arbitrarily small data can give rise to arbitrarily big solution
⇒ ill-posed problem

→ writing g arbitrarily, there is no guarantee that $\operatorname{supp} \tilde{f} \subseteq [0,1]$
T is not surjective, because $\tilde{f}$ from (*) not necessarily supported on $[0,1]$.

(This is quite common for convolution operators)