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Known reconstruction approaches (Sect 1.2)

- Tikhonov regularization (1.2.1)
- Iterative regularization (1.2.2)
- Regularization by discretization (1.2.3)

In all this methods trade between accuracy of the reconstruction and the closeness to an ill-posed problem. Now we'll state this concept in a more rigorous way:

1.3.3 <u>Deterministic regularization methods</u>

$$Tf = g^\delta \quad , T \in \mathcal{L}(X,Y), \quad R(T) \text{ not closed}$$

<u>Def 1.3.3 A</u> A pair $(\{R_\alpha\}_{\alpha\in A}, \bar{\alpha})$ with

- $\{R_\alpha\}_{\alpha\in A}$ a family of <u>continuous</u> mappings $R_\alpha : Y \to X$ with $R_\alpha(0) = 0$ (reconstruction operators)
- $A \mathrel{\hat=}$ parameter set with accumulation point 0.
- $\bar{\alpha} : \,]0,\infty[\, \times Y \to A$ (parameter choice rule) given g^δ and δ, $\bar{\alpha}$ tells us how strong the regularization we should use

is called a deterministic regularization method if

$$\lim_{\delta\to 0} \sup\left\{ \| R_{\bar{\alpha}(\delta,g^\delta)}\, g^\delta - T^+ g\|_X ,\ g^\delta \in Y : \|g^\delta - g\|_Y \le \delta \right\} = 0$$

for all $g \in \mathcal{D}(T^+)$

$\bar{\alpha}(\delta, g^\delta)$ concrete choice of regularization parameter

$\Rightarrow R_{\bar{\alpha}(\delta,g^\delta)}\, g^\delta \mathrel{\hat=}$ concrete solution

$\Rightarrow \|R_{\bar{\alpha}(\delta,g^\delta)}\, g^\delta - T^+ g\|_X \mathrel{\hat=}$ norm of total error

~~we take sup~~

$\Rightarrow \sup\{---\}$ from Def 1.1.3A is called worst case error (wcel

Tikhonov regularization + discrepancy principle:
($f_0 = 0$)

riconstr. operators $R_\alpha g^\alpha = (\alpha + T^*T)^{-1} T^* g$

param. choice rule discrepancy principle $\to$ later

$A = \mathbb{R}^+$

Negative results:

Theorem 1.3.3C If there is a deterministic regularization method $(R_\alpha, \bar{\alpha})$ for $T \in \mathcal{L}(X,Y)$ with $\bar{\alpha} = \bar{\alpha}(g^\delta)$, then $T^+ \in \mathcal{L}(X,Y)$ (*)

noise independependent parameter choice rule

$\Rightarrow$ for an ill-posed pb. we can never find a deterministic regularization method with ~~noise indep.~~ param. choice rule indep. of the noise level

Proof $g \in \mathcal{D}(T^+)$, sequence $(g_n)_n \subset \mathcal{D}(T^+)$: $g_n \to g$ in Y

from Def. 1.3.3A with $\delta = 0$, g_n replaces g^δ, g

$$R_{\bar{\alpha}(g_n)} g_n = T^+ g_n$$

from Def. 1.3.3 A with $\delta = \|g_n - g\|_Y$, and data g]

$$\|R_{\bar{\alpha}(g_n)} g_n - T^+ g\| \underset{\text{for } n \to \infty}{\longrightarrow} 0 \implies \|T^+ g_n - T^+ g\|_X \to 0 \text{ for } n \to \infty$$

$\Rightarrow$ since $(g_n)_n$ arbitrary $\Rightarrow T^+$ (sequentially) continuous $\Rightarrow T^+$ bound.

■

(*) $\mathcal{D}(T^+) = Y \Rightarrow R(T)$ closed $\Rightarrow T$ has bounded inverse.

Recall: For difference quotient reconstruction for differentiation in L^2:

$$\|R_{\bar{\alpha}} g^\delta - f\|_{L^2} = O(\sqrt{\delta}),$$

if $f \in C^1$, $f = g'$ $\leftarrow$ a priori knowledge

$\hat{=}$ convergence for $\delta \to 0$ with a rate (continuous function of δ which $\to 0$ as $\delta \to 0$)

Is this possible in the general case? NO:

Theorem 1.3.3D $(\{R_\alpha\}_{\alpha \in A}, \bar{\alpha})$ deterministic regularization method for $T \in \mathcal{L}(X,Y)$. If there exists a continuous function $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, $\varphi(0) = 0$ such that $wce(\delta) \le \varphi(\delta)$ for all $g \in \mathcal{D}(T^+)$, then $T^+ \in \mathcal{L}(Y,X)$

$\Downarrow$

rules out use of a priori information so no contradiction in the difference quotient reconstruction case.

Proof: $g \in \mathcal{D}(T^+)$, $(g_n)_n \subset \mathcal{D}(T^+)$

$g_n \to g$ in Y, $\delta_n = \|g_n - g\|_Y$

to show: $T^+(g_n) \to T^+(g)$

Note: $R_{\bar{\alpha}(\delta_n, g_n)} g^n \neq T^+ g_n$

$$\|T^+(g_n) - T^+(g)\|_X \le \underbrace{\|T^+(g_n) - R_{\bar{\alpha}(\delta_n, g_n)} g^n\|_X}_{w.c.e(\delta_n) \text{ for } g - g_n} + \underbrace{\|R_{\bar{\alpha}(g_n, \delta_n)} g^n - T^+ g\|}_{w.c.e(\delta_n) \text{ for } g}$$

$$\le 2\varphi(\delta_n) \underset{\text{as } n \to \infty}{\longrightarrow} 0 \quad (\text{since } \varphi \text{ continuous})$$ ■

$(\{R_\alpha\}_{\alpha\in A}, \bar{\alpha})$ d.r.m , $R(T)$ not closed (⇒ ill-posed pb.)

Assumption: 1) $\lim_{\alpha\to 0} R_\alpha(g) = T^+ g \quad \forall g \in \mathcal{D}(T^+)$ 2) R_α is linear, i.e. $R_\alpha \in \mathcal{L}(Y, X)$

<u>Theorem 1.3.3E</u>

(i) $(\|R_\alpha\|)_{\alpha\in A}$ is not bounded

(ii) $R_\alpha T$ does not converge in $\mathcal{L}(X)$ for $\alpha \to 0$.

<u>Proof</u>

(i) By Banach-Steinhaus ⊛ if $\|R_\alpha\| \le C \;\forall \alpha \in A$, then pointwise limit defines a bounded operator
⇒ contradiction because it would mean that T^+ is bounded

(ii) Assume $R_\alpha T \to \underbrace{T^+T}_{= P_{N(T)^\perp}}$ for $\alpha \to 0$ in $\mathcal{L}(Y)$
projection

$$\exists \alpha \in A : \|R_\alpha T - P_{N(T)^\perp}\|_{X\to X} \le \frac{1}{2} \qquad (*)$$

$$g \in \mathcal{D}(T^+) : \|T^+g\|_X \le \|\underbrace{T^+g}_{\in N(T)^\perp \Rightarrow T^+g = P_{N(T)^\perp}T^+g} - R_\alpha T T^+ g\|_X + \|R_\alpha \underbrace{TT^+g}_{P_{R(T)}g}\|_X$$

$$\le \frac{1}{2}\|T^+g\|_X + \|R_\alpha\|\|g\|_Y$$

$$\Rightarrow \|T^+g\| \le 2\|R_\alpha\|\|g\| \quad ⚡ \text{ because } T^+ \text{ is not bounded}$$

■

⊛ <u>Thm</u> (Banach-Steinhaus)

$(S_n)_n \in \mathcal{L}(X,Y)$, $\|S_n\| \le C \;\forall n$

$\lim_{n\to\infty} S_n x$ exists for all $x \in D \subset X$, D is dense in X.

Then $Tx = \lim_{n\to\infty} S_n x$ is bounded: $\|Tx\|_Y \le C\|x\|_X \;\forall x \in$