Inverse Problems

1.3.4 Singular value decomposition (SVD)

X, Y Hilbert spaces, $T \in K(X, Y)$
↳ compact operator

- Recall: SVD of matrices (LA)

$M \in \mathbb{C}^{n,m}$: $M = U \Sigma V^H$

where $U \in \mathbb{C}^{n,n}$, $V \in \mathbb{C}^{m,m}$, unitary, and

$$\Sigma = \operatorname{diag}(\sigma_1, \dots \sigma_{\min(m,n)}) \in \mathbb{R}^{m,n}$$

$n \geq m$: $\Sigma = \begin{pmatrix} \sigma_1 \sigma_2 \dots \sigma_m \\ 0 \end{pmatrix}$

$\sigma_i \hat{=}$ singular values, $\sigma_i \geq 0$

$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_{\min(m,n)} \geq 0$

$m \geq n$: $\Sigma = \begin{pmatrix} \sigma_1 \dots \sigma_n & 0 \end{pmatrix}$

assuming sorting, Σ is unique, but not U and V

$$\Rightarrow M = \sum_{\ell=1}^{\min(m,n)} \sigma_\ell \, \underline{u}_\ell \, \underline{v}_\ell^H, \qquad \left.\begin{aligned} \underline{u}_\ell &= U(:,\ell) \\ \underline{v}_\ell &= V(:,\ell) \end{aligned}\right\} \ell^{th} \text{ column}$$

$$M = \Big(\underbrace{\; U \;}_{r} \Big| \quad \Big) \left(\begin{matrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{matrix} \; \Sigma \; 0 \right) \left(\begin{matrix} V^H \\ \hline \end{matrix} \right) \}m-r$$

$\sigma_i \neq 0, i = 1, \dots r,$ $\quad r \leq \min(m,n)$; $\operatorname{rank}(M) = r$

the first r columns of U are an ONB of the image spa[ce]
the last $m-r$ rows of V^H are an ONB of kernel of M
the first r " " " " " of complement of kernel of M

$$\left.\begin{aligned} M^H M &= V \Sigma^H \Sigma V^H \\ M M^H &= U \Sigma \Sigma^H U^H \end{aligned}\right\} \Rightarrow \sigma_i^2 \text{ are eigenvalues of } MM^H, M^H M$$

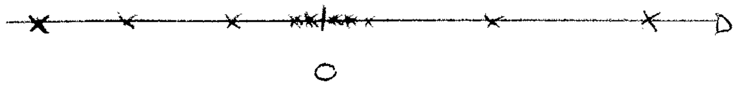
Pseudo-inverse $\quad M^+ = \sum_{\ell : \sigma_\ell > 0} \sigma_\ell^{-1} \, \underline{v}_\ell \, \underline{u}_\ell^H$ $\quad$ Moore-Penrose Pseudo-inverse

spans orth. complement of

hermitian matrices can be diagonali[zed] with uniform transformatio[n]

- Back to $T \in K(X,Y)$, $R(T)$ not closed

Thm (1.3.4 C): [Spectral thm for compact selfadjoint operators]
$A \in K(X)$, $A = A^*$. There exists an ONB $(f_j)_{j \in \mathbb{N}}$ of X st
$A f_j = \lambda_j f_j$ with eigenvalues $\lambda_j \in \mathbb{R}$ ↳ eigenvectors
and $A f = \sum_{j=1}^{\infty} \lambda_j (f, f_j) f_j$. 0 is the only accumulation point of (λ_j)



λ_j accumulate @ 0

Proof through Fredholm alternative

<u>Thm (1.3.4.D) SVD</u>

$T \in K(X,Y)$. There exist singular values $\sigma_1 \geq \sigma_2 \geq \ldots \geq 0$ and orthonormal bases $\{f_j\}_j \subset X$ $\{g_j\} \subset Y$ st

- $Tf = \sum_j \sigma_j (f, f_j)_X \, g_j$
- $\lim_{j \to \infty} \sigma_j = 0$

Terminology: $(\sigma_j, f_j, g_j) \mathrel{\hat{=}}$ singular system for T

<u>Proof</u>: Apply (1.3.4.C) to $A = T^*T \in K(X)$

$\Rightarrow$ eigenvector $(f_j)_j$, eigenvalues $(\lambda_j)_j \in \mathbb{R}$

$$\lambda_j = (\lambda_j f_j, f_j)_X = (T^*T f_j, f_j)_X = \| T f_j \|_Y^2 \geq 0$$

$$T^*T f = \sum \lambda_j (f, f_j)_X \, f_j$$

Define: $\sigma_j = \sqrt{\lambda_j} = \| T f_j \|_Y$

$$g_j = \frac{T f_j}{\sigma_j} \quad , \text{if } T f_j \neq 0$$

f is ONB $\Rightarrow$ $f = \sum (f, f_j) f_j$

$$Tf = \sum_j (f, f_j) \, T f_j = \sum_j \sigma_j (f, f_j) \, g_j$$

$$(g_j, g_e)_Y = \frac{(T f_j, T f_e)_Y}{\sigma_j \sigma_e} = \frac{(T^*T f_j, f_e)_X}{\sigma_j \sigma_e} = \frac{\lambda_j}{\sigma_j \sigma_e} (f_j, f_e)_X = \delta_{ij}$$

$\Rightarrow \{g_j\}$ is an ON system $\Rightarrow$ can be made into an ONB by Hahn-Banach

□

SVD and generalized inverses:

<u>Lemma 1.3.4.E:</u> (Picard condition)

$T \in K(X,Y)$ with singular system $(\sigma_j, f_j, g_j)_j$

$$g \in D(T^+) \underset{\text{Thm } 1.3.1.G}{\overset{}{=}} R(T) + R(T)^\perp = \Big\{ g \in Y : \sum_j \sigma_j^{-2} \, |\underbrace{(g, g_j)_Y}_{\text{"Fourier coefficients"}}|^2 < \infty \Big\}$$

$\Rightarrow$ Decay of "Fourier coefficients" of $g \in D(T^+)$ has to overcompensate blow-up of σ_j^{-1}

Proof (i) $g \in \mathcal{R}(T)^\perp \iff (g, g_j)_Y = 0 \quad \forall j$ st $g_j = \frac{Tf_j}{\sigma_j}$

(ii) $g \in \mathcal{R}(T) \iff g = Tf \;, f \in X$

"$\Rightarrow$" $\sum_j \sigma_j^{-2} |(Tf, g_j)_Y|^2 \underset{\uparrow}{=} \sum_j \sigma_j^{-2} |(f, \sigma_j f_j)|^2 = \|f\|_X^2$

$\left[g_j = \sigma_j^{-1} T f_j \Rightarrow T^* g_j = \sigma_j^{-1} T^* T f_j = \sigma_j f_j\right]$

"$\Leftarrow$" $f = \sum_j \sigma_j^{-1} (g, g_j)_Y f_j \in X$ sum exists

$\underset{\substack{\nearrow \\ Tf_j = \sigma_j g_j}}{Tf} = \sum_j \sigma_j^{-1} (g, g_j)_Y \sigma_j g_j \underset{\uparrow}{=} g$

ONB property of $(g_j)_j$

$\Rightarrow g \in \mathcal{R}(T)$ $\square$

Thm 1.3.4.F: $T \in K(X,Y)$ with singular system (σ_j, f_j, g_j)

$\Rightarrow T^+ g = \sum_{j : \sigma_j > 0} \sigma_j^{-1} (g, g_j) f_j \quad \forall g \in \mathcal{D}(T^+)$

Sketch of proof: $TT^+ g = \sum_{j, \sigma_j > 0} (g, g_j)_Y g_j = P_{\overline{\mathcal{R}(T)}} g$ $\square$

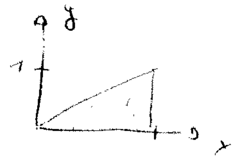
T^+ must be unbounded, $\sigma_j \to 0 \Rightarrow T^+$ unbounded

" Speed of decay of σ_j quantifies the ill-posedness "

Examples

$X = Y = L^2(0,1)$: $Tf = \int_0^x f(\xi)\, d\xi \;, \; 0 \le x \le 1$

$(Tf, g)_{L^2} = \int_0^1 \int_0^x f(\xi)\, d\xi \; g(x)\, dx =$



$= \int_0^1 \int_\xi^1 g(x)\, dx \; f(\xi)\, d\xi$

$\Rightarrow T^* g(\xi) = \int_\xi^1 g(x)\, dx$

$(T^* T f)(x) = \int_x^1 \int_0^\xi f(\tau)\, d\tau\, d\xi \overset{!}{=} \lambda f(x) \quad , \quad \lambda \neq 0$

$\frac{d}{dx}$: $- \int_0^x f(\tau)\, d(\tau) = \lambda f'(x)$

$\frac{d}{dx}$: $- f(x) = \lambda f''(x)$

$\Rightarrow f(x) = A \cos\left(\frac{1}{\sqrt{\lambda}} x\right) + B \sin\left(\frac{1}{\sqrt{\lambda}} x\right)$

Boundary condition: $f(1) = 0$, $f'(0)$

$$\frac{1}{\sqrt{\lambda}} \in (\mathbb{Z} + 1/2)\pi \qquad B = 0$$

$\Rightarrow f_j$, $\lambda > 0 \Rightarrow \frac{1}{\sqrt{\lambda}} \in (\mathbb{N}_0 + 1/2)\pi$

$$\left.\begin{aligned} &\Rightarrow f_j(x) = \sqrt{2}\cos\left((j - 1/2)\pi x\right), \quad j \in \mathbb{N} \\ &\Rightarrow g_j(x) = \sqrt{2}\sin\left(\left(j - \tfrac{1}{2}\right)\pi x\right) \end{aligned}\right\} \text{"Fourier singular system"}$$

$$\sigma_j = \frac{1}{(j - 1/2)\pi} \to 0 \qquad j \to \infty$$