Boundary condition: $f(1) = 0$, $f'(0)$

$$\downarrow \qquad\qquad \downarrow$$

$$B = 0$$

$$\frac{1}{\sqrt{\lambda}} \in (\mathbb{Z} + 1/2)\pi$$

$\Rightarrow f_j$, $\lambda > 0 \Rightarrow \frac{1}{\sqrt{\lambda}} \in (\mathbb{N}_0 + 1/2)\pi$

$\Rightarrow f_j(x) = \sqrt{2}\cos\left((j - 1/2)\pi x\right)$, $j \in \mathbb{N}$

$\Rightarrow g_j(x) = \sqrt{2} \quad \sin\left((j - \frac{1}{2})\pi x\right)$ } "Fourier singular system"

$\sigma_j = \frac{1}{(j - 1/2)\pi} \rightarrow 0 \qquad j \rightarrow \infty$

polynomial decay of sing. values $\rightarrow$ <u>mildly ill-posed</u>

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Example: Backward heat equation $X = Y = L^2(0,1)$

$$\partial_t u = \partial_{xx} u \qquad \text{on }]0,1[\times]0,T[$$

$$u(0,t) = u(1,t) = 0 \qquad \forall\ 0 \le t \le T$$

Data: $u(x, T)$, $T > 0$ fixed

Unknown: $u(x, 0)$

Forward operator: $(Tf)(x) = \int_0^1 \underbrace{2\left\{\sum_{n=1}^{\infty} e^{-n^2\pi^2 T} \sin(n\pi x)\sin(n\pi y)\right\}}_{\text{kernel } k(x,y) \in L^2(]0,1[^2)} f(y)\, dy$

$k(x,y) = k(y,x) \Rightarrow T$ is self-adjoint and compact (kernel in L^2)

$\Rightarrow f_j = g_j$ and $\sigma_j \hat{=}$ eigenvalues of T (correspond to)

SVD = spectral decomposition

$$Tf = \sum_{j=1}^{\infty} \lambda_j (f, f_j) \underset{\text{eigenfunctions}}{f_j}$$

concrete $(Tf)(x) = \sum_{n=1}^{\infty} \underbrace{e^{-n^2\pi^2 T}}_{\text{eigenvalues}} \underbrace{\left(\int_0^1 \sqrt{2}\sin(\pi n y) f(y)\, dy\right)}_{(f, f_j)} \underbrace{\sqrt{2}\sin(\pi n x)}_{f_j}$

Here: $\underbrace{\sigma_j = e^{-j^2\pi^2 T}}$, $f_j = g_j = \sqrt{2}\sin(\pi j x)$

more than exponential decay $\rightarrow$ <u>severly ill-posed</u>

1.3.5 Convergence of linear regularization method

$\{R_\alpha\}_{\alpha \in \mathcal{A}} \subset \mathcal{L}(X, Y)$

1.3.5.1 Reconstruction by filters

Singular system for $T \in K(X, Y)$ given by $(\sigma_j, f_j, g_j)_{j=1}^{\infty}$

$\mathcal{A} \mathrel{\hat{=}}$ parameter set with accumulation point 0, e.g. $\mathcal{A} = \mathbb{R}^+$

$\Rightarrow$ Given $\{q_\alpha\}_{\alpha \in \mathcal{A}}$, $q_\alpha : \mathbb{R}_0^+ \to \mathbb{R}$, define

$$R_\alpha g^\delta = \sum_{j=1}^{\infty} q_\alpha(\sigma_j^2) (T^* g^\delta, f_j) f_j \qquad g^\delta \in Y$$

Recall: "functonal calculus"

$T \in K(X)$, $T = T^*$, $\phi : \mathbb{R} \to \mathbb{R}$, ϕ bounded

$$\underbrace{\phi(T)}_{\in \mathcal{L}(X)} f = \sum \underbrace{\phi(\lambda_j)}_{\text{eigenvalues}} (f, f_j) \underbrace{f_j}_{\text{eigenfunctions}}$$

$A \in \mathbb{C}^{n,n}$, $A = A^H \Rightarrow A = U D U^H$

$\exp(A) = U \exp(D) U^H$

$\phi(A) = U \left(\phi(d_1) \cdots \phi d_n \right) U^H$

$\Rightarrow \quad R_\alpha = q_\alpha(T^*T)\, T^*$

Approximation error: $g = Tf^+$

$$f^+ - R_\alpha T f^+ = (Id - q_\alpha(\underline{T^*T})\, T^*T) f^+$$

$$= r_\alpha(T^*T) f^+$$

with $r_\alpha(\lambda) = 1 - \lambda q_\alpha(\lambda)$

<u>Def 1.3.5.1 (Filter)</u> A family of measurable functions $q_\alpha : \mathbb{R}_0^+ \to \mathbb{R}$, $\alpha \in \mathcal{A}$, is a filter, if

(F1) $\lim_{\alpha \to 0} r_\alpha(\lambda) = \begin{cases} 0 & , \text{for } \lambda > 0 \\ 1 & , \lambda = 0 \end{cases}$

(F2) $\exists\, C_r > 0 : |r_\alpha(\lambda)| \leq C_r \quad \forall \alpha \in \mathcal{A}, \lambda \geq 0$, where $r_\alpha(\lambda) = 1 - \lambda q_\alpha(\lambda)$

(F3) $\sqrt{\lambda} |q_\alpha(\lambda)| \leq \frac{C_q}{\sqrt{\alpha}}$ for some $C_q > 0$ and all $\alpha \in \mathcal{A}, \lambda \geq 0$

(F1) $\Rightarrow \lim_{\alpha \to 0} q_\alpha(\lambda) = \frac{1}{\lambda}$, $\lambda > 0$ [Blow-up is necessary!]

Example. Tikhonov regularization ($f_0 = 0$)

$$R_\alpha g^\delta := \operatorname{argmin}_f \left\{ \| g^\delta - Tf \|_Y^2 + \alpha \| f \|_X^2 \right\}$$

$$\Leftrightarrow \quad R_\alpha = (T^*T + \alpha \, \mathrm{Id})^{-1} T^* \quad , \alpha > 0$$

$$\Rightarrow \quad R_\alpha = q_\alpha(T^*T)\, T^* \quad \text{with} \quad q_\alpha(\lambda) = \frac{1}{\lambda + \alpha}$$

$$\Rightarrow \quad r_\alpha(\lambda) = 1 - \frac{\lambda}{\lambda+\alpha} = \frac{\alpha}{\lambda+\alpha} \quad \Rightarrow \quad (F1) \checkmark$$

(F2) satisfied with $C_r = 1$

(F3) $$\alpha \lambda \, q_\alpha(\lambda)^2 = \alpha \lambda \frac{1}{(\lambda+\alpha)^2} = \frac{\alpha\lambda}{\underbrace{\lambda^2}_{\geq 0} + 2\alpha\lambda + \underbrace{\alpha^2}_{\geq 0}} \leq \frac{\alpha\lambda}{2\alpha\lambda} = \frac{1}{2}$$

Example: Landweber

$$f_0^+ := 0 \qquad f_{k+1}^+ := f_k^+ - \mu \, T^*(T f_k^+ - g^\delta) \,, \quad \mu \leq \| T^*T \|_X$$

$$\alpha = \frac{1}{j} : \quad R_\alpha g^\delta := f_j^+ = \sum_{k=0}^{j-1} (\mathrm{Id} - \mu T^*T)^k T^* g^\delta$$

$$\mathcal{A} = \{ 1/j \}_{j \in \mathbb{N}}$$

wlog: $\mu = 1$ ($\| T^*T \| \leq 1$) $\qquad R_\alpha = q_\alpha(T^*T)\, T^*$ with

can always be achieved rescaling the norm $\| \; \|_X$

$$q_\alpha(\lambda) = \sum_{k=0}^{j-1} (\mathrm{Id} - \lambda)^k \overset{\lambda \in (0,1)}{=} \frac{1 - (1-\lambda)^j}{\lambda}$$

$$\Rightarrow \quad r_\alpha(\lambda) = (1-\lambda)^j$$

- (F1): here $j \to \infty$ ✓
- (F2): $C_r = 1$
- (F3): $\cancel{\frac{\lambda}{j} \left[\frac{1 - (1-\lambda)^j}{\lambda} \right]^2 = \frac{1}{j\lambda} \left(1 - 2(1-\lambda)^j + (1-\lambda)^{2j} \right)}$

we know $\quad \lambda \, |q_\alpha(\lambda)| \leq 1$

$\qquad\qquad |q_\alpha(\lambda)| \leq j \qquad \left(q_\alpha(\lambda) = \sum_{k=0}^{j-1} (1-\lambda)^k \right)$

multiply $\quad \lambda \, |q_\alpha(\lambda)|^2 \leq j = \frac{1}{\alpha}$

$\Rightarrow \quad C_q = 1$

Example: Regularization by truncated SVD

Motivated by: $T^+ g = \sum_{j:\sigma_j \neq 0} \sigma_j^{-1} (g, g_j) f_j$

$$\to R_\alpha g := \sum_{j:\sigma_j^2 > \alpha} \sigma_j^{-1} (g, g_j) f_j \;, \alpha > 0$$

Recall: $g_j = \frac{1}{\sigma_j} T f_j$

$$(g, g_j)_Y = (g, \tfrac{1}{\sigma_j} T f_j)_Y = (\tfrac{1}{\sigma_j} T^* g, f_j)_X$$

Related filter

$$R_\alpha g = \sum_{j:\sigma_j^2 > \alpha} \frac{1}{\sigma_j^2} (T^* g, f_j) f_j$$

$$q_\alpha(\lambda) = \frac{1}{\lambda} \chi_{[\alpha, \infty]}(\lambda)$$

$$r_\alpha(\lambda) = 1 - \lambda q_\alpha(\lambda) = \chi_{[0,\alpha]}(\lambda)$$

(F1) - (F2) - (F3) with $c_r = 1$, $c_q = 1$

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$T \in K(X,Y)$ with singular system $\{\sigma_j, f_j, g_j\}$

Reconstruction operator induced by filter fct $q_\alpha : \mathbb{R}^+ \to \mathbb{R}$

$$R_\alpha g := q_\alpha(T^* T) T^* g = \sum_{j=1}^{\infty} q_\alpha(\sigma_j^2) (T^* g, f_j) f_j$$

Approximation error:

$$f - R_\alpha T f = f - q_\alpha(T^* T) T^* T f = r_\alpha(T^* T) f$$

with $r_\alpha(\lambda) = 1 - \lambda q_\alpha(\lambda)$

residual: $T R_\alpha g^\delta - g^\delta = \sum_k \sigma_k (R_\alpha g^\delta, f_k)_X g_k - (g^\delta, g_k)_Y g_k$

$$= \sum_k \sigma_k q_\alpha(\sigma_k^2) (T^* g^\delta, f_k)_X g_k - (g^\delta, g_k)_Y g_k$$

$[T f_k = \sigma_k g_k]$

$$= \sum_k \underbrace{(\sigma_k^2 q_\alpha(\sigma_k^2) - 1)}_{-r_\alpha(\sigma_k^2)} (g^\delta, g_k)_Y g_k \qquad [\sigma_k^2 \text{ EV of } TT^*$$

$\boxed{TT^* \text{ its}}$ because g_k are eigenfcts

$$= -r_\alpha(TT^*) g$$