Example: Regularization by truncated SVD

Motivated by: $T^+ g = \sum_{j:\sigma_j \neq 0} \sigma_j^{-1} (g, g_j) f_j$

$$\rightarrow R_\alpha g := \sum_{j:\sigma_j^2 > 0} \sigma_j^{-1} (g, g_j) f_j \quad , \alpha > 0$$

Recall: $g_j = \frac{1}{\sigma_j} T f_j$

$$(g, g_j)_Y = (g, \frac{1}{\sigma_j} T f_j)_Y = (\frac{1}{\sigma_j} T^* g, f_j)_X$$

Related filter

$$R_\alpha g = \sum_{j:\sigma_j^2 > \alpha} \frac{1}{\sigma_j^2} (T^* g, f_j) f_j$$

$$q_\alpha(\lambda) = \frac{1}{\lambda} \chi_{[\alpha, \infty]}(\lambda)$$

$$r_\alpha(\lambda) = 1 - \lambda q_\alpha(\lambda) = \chi_{[0,\alpha]}(\lambda)$$

(F1) - (F2) - (F3) with $c_r = 1$, $c_q = 1$

$T \in K(X, Y)$ with singular system $\{\sigma_j, f_j, g_j\}$

Reconstruction operator induced by filter fct $q_\alpha : \mathbb{R}^+ \rightarrow \mathbb{R}$

$$R_\alpha g := q_\alpha (T^* T) T^* g = \sum_{j=1}^{\infty} q_\alpha(\sigma_j^2) (T^* g, f_j) f_j$$

Approximation error:

$$f - R_\alpha T f = f - q_\alpha(T^* T) T^* T f = r_\alpha(T^* T) f$$

with $r_\alpha(\lambda) = 1 - \lambda q_\alpha(\lambda)$

residual: $$T R_\alpha g^\delta - g^\delta = \sum_k \sigma_k (R_\alpha g^\delta, f_k)_X g_k - (g^\delta, g_k)_Y g_k$$

$$= \sum_k \sigma_k q_\alpha(\sigma_k^2) (T^* g^\delta, f_k)_X g_k - (g^\delta, g_k)_Y g_k$$

$[T f_k = \sigma_k g_k]$

$$= \sum_k \underbrace{(\sigma_k^2 q_\alpha(\sigma_k^2) - 1)}_{-r_\alpha(\sigma_k^2)} (g^\delta, g_k)_Y g_k$$

[σ_k^2 EV of TT^* — TT^* because g_k are its eigenfcts]

$$= - r_\alpha(T T^*) g$$

Filter properties: (F1) $\lim_{\alpha \to 0} q_\alpha(\lambda) = \frac{1}{\lambda}$ for $\lambda > 0$

(equivalent to $\lim_{\alpha \to 0} r_\alpha = \begin{cases} 1 & \lambda = 0 \\ 0 & \lambda > 0 \end{cases}$

(F2) $|r_\alpha(\lambda)| \le C_r$

(F3) $\sqrt{\lambda}|q_\alpha(\lambda)| \le C_q/\sqrt{\alpha}$

Tikhonov: $q_\alpha(\lambda) = \frac{1}{\alpha + \lambda}$ Landweber: $q_\alpha(\lambda) = \frac{1}{\lambda}\left(1 - (1-\lambda)^j\right)$, $\alpha = \frac{1}{j}$

truncated SVD: $q_\alpha(\lambda) = \frac{1}{\lambda} \chi_{[\alpha^2, \infty)}(\lambda)$

<u>Lemma 1.3.5.B</u> $\{q_\alpha\}_{\alpha \in A}$ filter, Def. 1.3.5.A

$\Rightarrow$ (i) $\lim_{\alpha \to 0} R_\alpha g = T^+ g \quad \forall g \in D(T^*)$

(ii) $\|R_\alpha\| \le C_q/\sqrt{\alpha}$

<u>Proof:</u> (i) $g \in D(T^+)$, $f^+ := T^+ g \in N(T)^\perp$

$$r_0(T^*T) f = \sum_{j,\, \sigma_j = 0} (f, f_j)_X f_j = P_{N(T)} f$$

$$\Rightarrow r_0(T^*T) f^+ = 0 \qquad \underbrace{Tf^+ = TT^+ g = P_{\overline{R(T)}}\, g}$$

$$T^+ g - R_\alpha g = f^+ - R_\alpha g = f^+ - R_\alpha T f^+ + \underbrace{R_\alpha P_{R(T)^\perp} g}_{=0,\ \text{because}}$$

$$= r_\alpha(T^*T) f^+$$

$\left(R_\alpha g = \sum_j q_\alpha(\sigma_j^2) \underbrace{(g, Tf_j)_Y}_{=0} f_j \right)$

$$\|(r_\alpha(T^*T) - r_0(T^*T)) f^+\|_X^2 = \sum |r_\alpha(\sigma_j^2) - \underbrace{r_0(\sigma_j^2)}_{=0 \text{ if } \sigma_j \neq 0}|^2 \underbrace{|(f^+, f_j)_X|^2}_{=0 \text{ if } \sigma_j = 0}$$

$$= \sum_{j,\, \sigma_j > 0} |r_\alpha(\sigma_j^2)|^2 |(f^+, f_j)|^2$$

Fix $\varepsilon > 0$: $\exists J_\varepsilon \in \mathbb{N}$: $\sum_{j > J_\varepsilon} C_r^2 |(f^+, f_j)_X|^2 < \frac{\varepsilon}{2}$

$\lim_{\alpha \to 0} r_\alpha(\sigma_j^2) = 0$: $\exists \alpha_\varepsilon$: $|r_\alpha(\sigma_j)| < \varepsilon/2 \quad \forall j = 1, \dots J_\varepsilon$

$$\le \sum_{j < J_\varepsilon} (\varepsilon/2) |(f^+, f_j)|^2 + \varepsilon/2 \le \varepsilon \|f^+\|_X^2 \qquad (\text{and } \varepsilon \text{ arbitrary small})$$

(ii) General $\|F(T)\| \le \sup_{\lambda \in \sigma(T)} |F(\lambda)|$

$$\|R_\alpha\|^2 = \|R_\alpha^* R_\alpha\| = \|q_\alpha(T^*T) T^*T\, q_\alpha(T^*T)\|$$

$$\le \sup_{0 \le \lambda \le \|T\|^2} q_\alpha^2(\lambda)\lambda \underset{F3}{\le} C_q^2/\alpha \qquad \square$$

$g \in D(T^+)$, $\| g - g^\delta \|_Y \le \delta$

$$\| T^+ g - R_\alpha g^\delta \|_X \le \underbrace{\| T^+ g - R_\alpha g \|_X}_{\to 0 \text{ as } \alpha \to 0 \ (\text{PCR1})} + \underbrace{\| R_\alpha (g - g^\delta) \|_X}_{\le (C_q/\sqrt{\alpha})\,\delta \ (\text{PCR2})}$$

<u>Thm 1.3.5.C</u> $\{R_\alpha\}_{\alpha \in A}$ induced by a filter $\{q_\alpha\}_{\alpha \in A}$.
If $\bar\alpha : \mathbb{R}_0^+ \times Y \to A$, $\bar\alpha = \bar\alpha(\delta, g^\delta)$ is a parameter choice rule with

(PCR1) $\sup\{ \bar\alpha(\delta, g^\delta) : g^\delta \in Y : \| g - g^\delta \|_Y \le \delta \} \to 0$

(PCR2) $\sup\{ \delta / \sqrt{\bar\alpha(\delta, g^\delta)} :$ " " $\} \to 0$

as $\delta \to 0$

for all $g \in D(T^+)$. Then $(R_\alpha, \bar\alpha)_{\alpha \in A}$ is a deterministic regularization method (Def 1.3.3.A)

1.3.5.2 <u>Spectral source condition</u> (a priori knowledge)

Hölder / Sobolev source condition:

$$f^+ \in X_\nu := \{ f \in X : \sum_{j=1}^{\infty} \underbrace{\sigma_j^{-2\nu}}_{\to \infty \text{ as } j \to \infty} |(f, f_j)|^2 < \infty \}, \quad \nu \ge 0$$

↗ exact solution

(a priori knowledge defined through operator
in practice very difficult to check

Relationship with Sobolev spaces:

$L^2_{per}([0, 2\pi]) =: X$

$f_j(\zeta) = \frac{1}{2\pi} e^{-ij\zeta}$: ONB of X

$$H^m([0, 2\pi]) = \{ f : \sum_{j \in \mathbb{Z}} (1 + |j|)^{2m} |(f, f_j)_X|^2 < \infty \}$$

Notation: $\| f \|_\nu^2 := \sum_{j=1}^{\infty} \sigma_j^{-2\nu} |(f, f_j)|^2$

Example: $(Tf)(x) = \int_0^x f(t)\, dt$, $0 \le x \le 1$, $X = L^2(0,1)$

$\to$ singular system $\left\{ \frac{1}{\pi(j - 1/2)}, \sqrt{2} \cos((j - 1/2)\pi x), \sqrt{2} \sin((j - 1/2)\pi x) \right\}$

Known: $\left(\int_0^1 \varphi(\tau) \cos((j - 1/2)\pi\tau)\, d\tau \right)_{j \in \mathbb{N}}$ is square summable

$\Leftrightarrow \varphi \in L^2(0,1)$

$\Rightarrow X_1 = \{ f \in L^2(0,1), f' \in L^2(0,1), f(1) = 0 \} \subset H^1(0,1)$

$X_2 = \{ f \in L^2(0,1), f' \in L^2(0,1), f'' \in L^2(0,1), f(0) = f'(0) = 0 \} \subset H^2(0,2)$

1.3.5.3 Optimality

Def 1.3.5.E worst case error

$$\mathrm{wce}(\delta, K, R) := \sup \left\{ \|f^+ - Rg^\delta\|_X : g^\delta \in K,\ f^+ \in K,\ \|Tf^+ - g^\delta\|_Y \le \delta \right\}$$

- δ: noise level
- K: $\subset X$, reflects a priori knowledge
- R: reconstruction operator: $R : Y \to X$ continuous, $R(0) = 0$

best worst case error

$$\mathrm{wce}(\delta, K) := \inf_R \ \mathrm{wce}(\delta, K, R) \quad \left(\begin{array}{l}\text{inf. over any}\\ \text{reconstruction operator}\end{array}\right)$$

Now: $K = K_{\nu,\varrho} := \{ f \in X_\nu : \|f\|_\nu \le \varrho \}$

$\{R_\alpha\}_{\alpha \in \mathcal{A}}$ induced by filter $\{q_\alpha\}_{\alpha \in \mathcal{A}}$, $R_\alpha \in \mathcal{L}(X, Y)$

$$\|f^+ - R_\alpha g^\delta\| \le \underbrace{\|f^+ - R_\alpha T f^+\|_X}_{} + \underbrace{\|R_\alpha (Tf^+ - g^\delta)\|_X}_{\text{Lemma 1.3.5.B } \le (C_q/\sqrt{\alpha})\,\delta}$$

$$\|f^+ - R_\alpha T f^+\|_X^2 = \sum_{j, \sigma_j > 0} |r_\alpha(\sigma_j^2)|^2 \underbrace{|(f^+, f_j)|^2}_{\sigma^{2\nu} \cdot \sigma_j^{-2\nu}} \le \sup_{j, \sigma_j > 0} \sigma_j^{2\nu} |r_\alpha(\sigma_j^2)|^2 \cdot \underbrace{\sum_{j, \sigma_j > 0} \sigma_j^{-2\nu} |(f^+, f_j)_X|^2}_{= \|f^+\|_\nu^2}$$

$f^+ \in K_{\nu,\varrho}$

> We'll see:
> Needed: Qualification condition
>
> $$\exists\, C_\nu > 0 : \sup_{0 \le \lambda \le \|T\|^2} \lambda^{\nu/2} |r_\alpha(\lambda)| \le C_\nu \sqrt{\alpha} \quad \forall \alpha \in \mathcal{A} \qquad (1.3.5F)$$
>
> $\Rightarrow \|f^+ - R_\alpha T f^+\|_X^2 \le C_\nu^2\, \alpha\, \varrho^2$

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Apriori knowledge $f^+ \in K_{\nu,\varrho} := \{ f \in X : \|f\|_\nu \le \varrho \}$, where

$$\|f\|_\nu^2 = \sum \sigma_j^{-2\nu} |(f, f_j)_X|^2$$

Using $f^+ \in K_{\nu,\varrho}$:

$$\|f^+ - R_\alpha T f^+\|_X^2 = \sum_j \sigma_j^{2\nu} |r_\alpha(\sigma_j^2)|^2\, \sigma_j^{-2\nu} |(f^+, f)_X|^2$$

$$\le \max_{0 \le \lambda \le \|T\|^2} \lambda^\nu |r_\alpha(\lambda)|^2\, \|f^+\|_\nu^2$$

Assumption 1.3.5.F: (Qualification assumption)

$$\exists\, C_\nu > 0 \quad \lambda^{\nu/2} |r_\alpha(\lambda)| \le C_\nu\, \alpha^{1/2} \quad \forall \lambda, \alpha$$

If this holds for all $\nu \in [0, \nu_0]$, we say that the filter has qualification ν_0

With Q.A. 1.3.5.F:

$$\| f^+ - R_\alpha T f^+ \|_X^2 \le C_\nu^2 \alpha^\nu \rho^2$$

Lemma 1.3.5.B

Data noise error: $\| R_\alpha T f^+ - R_\alpha g^\delta \| \le \| R_\alpha \| \, \| T f^+ - g^\delta \| \le \frac{C_q}{\sqrt{\alpha}} \delta$

Total reconstruction error:

$$\| f^+ - R_\alpha g^\delta \|_X \le \underbrace{C_\nu \alpha^{\nu/2} \rho + C_q \alpha^{-1/2} \delta}_{\text{minimize this w.r.t. } \alpha}$$

$$\left.\begin{aligned} \varphi(t) &= A t^{\nu/2} + B t^{-1/2} \\ \varphi'(t) &= \frac{\nu}{2} A t^{\nu/2 - 1} + \frac{B}{2} t^{-3/2} \overset{!}{=} 0 \\ \Rightarrow \quad t^{\nu/2 + 1/2} &= \frac{B}{\nu A} \end{aligned}\right\} \quad \alpha_{opt} = \left(\frac{C_q \delta}{\nu C_\nu \rho} \right)^{2/\nu+1}$$

$$\| f^+ - R_{\alpha_{opt}} g^\delta \|_X \le C_\nu \left(\frac{C_q \delta}{\nu C_\nu \rho} \right)^{\nu/\nu+1} \rho + C_q \left(\frac{C_q \delta}{\nu C_\nu \rho} \right)^{-1/\nu+1} \delta^{\frac{\nu+1}{\nu+1}}$$

$$\le \underbrace{C_\nu^{\frac{1}{\nu+1}} C_q^{\frac{\nu}{\nu+1}} \left(\nu^{-\frac{\nu}{\nu+1}} + \nu^{\frac{1}{\nu+1}} \right)}_{\text{constant}} \delta^{\frac{\nu}{\nu+1}} \rho^{\frac{1}{\nu+1}}$$

rate of crg. of reconstruction error

Thm 1.3.5.G: Assume $f \in K_{\nu,\rho}$ and $\{q_\alpha\}_{\alpha \in A}$ a filter with qualification $\nu_0 \ge \nu$. Then, for

$$\alpha \approx \left(\delta / \rho \right)^{2/\nu+1} =: \alpha_{opt} \quad (*)$$

$$wce(\delta, K_{\nu,\rho}, R_{\alpha_{opt}}) \le C(\nu) \, \delta^{\nu/\nu+1} \rho^{1/\nu+1}$$

with $C(\nu) > 0$ independent of ρ, δ

(*) $\hat{=}$ a priori parameter choice rule

Recall Thm 1.3.5.C: $\Rightarrow$ $(R_\alpha, \bar{\alpha} = \alpha_{opt}(\delta))$ is a deterministic regularization method

$$wce(\delta, K_{\nu,\rho}) = \inf_R \sup \{ \| f^+ - R g^\delta \|_X : f^+ \in K_{\nu,\rho}, \| g^\delta - T f^+ \|_Y \le \delta \}$$

$$\underset{g^\delta = 0}{\ge} \underbrace{\sup \{ \| f^+ \|_X : f^+ \in K_{\nu,\rho}, \| T f^+ \| \le \delta \}}_{=: \, w(\delta, K_{\nu,\rho})}$$

$\hookrightarrow$ with $K_{\nu,\rho}$ replaced with $B_1(0)$ in X

Thm 1.3.5.H $\quad \omega(\delta, K_{\nu,\varrho}) \le \delta^{\nu/(\nu+1)} \varrho^{1/(1+\nu)}$

and the rate w.r.t. $\delta \to 0$ is sharp

Proof: $\|f^+\|_X^2 = \sum_j |(f, f_j)_X|^2$

Hölder ineq $\quad \sum_{j=1}^{\infty} x_j y_j \le \left(\sum_j x_j^p\right)^{1/p} \left(\sum_j y_j^q\right)^{1/q}$, $\quad 1 \le p,q < \infty$, $\frac{1}{p}+\frac{1}{q}=1$

$x_j, y_j \ge 0$

with $q = \nu+1$, $p = \frac{\nu+1}{\nu}$

$$\|f^+\|_X^2 = \sum_j \underbrace{\sigma_j^{2/p} |(f^+, f_j)_X|^{2/p}}_{x_j} \underbrace{\sigma_j^{-2/p} |(f^+, f_j)_X|^{2/q}}_{y_j}$$

$$\le \left(\sum \sigma_j^2 |(f, f_j)_X|^2\right)^{1/p} \cdot \left(\sum \sigma_j^{-2q/p} |(f^+, f_j)_X|^2\right)^{1/q}$$

$\left[Tf^+ = \sum \sigma_j (f^+, f_j)_X f_j\right] \qquad [q/p = \nu]$

$$\le \underbrace{\|Tf^+\|_Y^{2/p}}_{\le \delta} \cdot \underbrace{\|f^+\|_\nu^{2/q}}_{\le \varrho} \le \left(\delta^{\nu/(\nu+1)}\right)^2 \left(\varrho^{1/(\nu+1)}\right)^2$$

"Sharpness" (incomplete)

$\delta_k := \varrho \sigma_k^{\nu+1} \xrightarrow[k \to \infty]{} 0$

$f_k^+ := \varrho \sigma_k^\nu f_k \qquad \|f_k^+\|_\nu = \varrho$

$Tf_k^+ = \varrho \sigma_k^{\nu+1} g_k \qquad \|Tf_k^+\|_Y = \varrho \sigma_k^{\nu+1} = \delta_k$

$\omega(\delta_k, K_{\nu,\varrho}) \ge \|f_k^+\|_X = \varrho \sigma_k^\nu = \varrho \left(\delta_k/\varrho\right)^{\nu/(\nu+1)} = \delta_k^{\nu/(\nu+1)} \varrho^{1/(\nu+1)}$ $\quad \square$

$\Rightarrow$ Using thm 1.3.5.C:

$$\delta^{\frac{\nu}{\nu+1}} \varrho^{\frac{1}{\nu+1}} \le wce(\delta, K_{\nu,\varrho}) \le wce(\delta, K_{\nu,\varrho}, R_{\alpha_{opt}}) \le C \delta^{\nu/(\nu+1)} \varrho^{1/(\nu+1)}$$

Qualifications

- Tikhonov $\quad r_\alpha(\lambda) = \frac{\alpha}{\lambda+\alpha}$

$\lambda^{\nu/2} \frac{\alpha}{\lambda+\alpha} \le C_\nu \alpha^{\nu/2}$

$\xi = \lambda/\alpha$: $\quad \xi^{\nu/2} \frac{1}{\xi+1} \le C_\nu \quad \forall \xi$

$\Leftrightarrow \quad \nu \le 2$

$\to$ qualification $\nu_0 = 2$

- Landweber $\quad r_\alpha(\lambda) = (1-\lambda)^j$, $\alpha = \frac{1}{j}$, $j \in \mathbb{N}$

$\lambda^{\nu/2}(1-\lambda)^j \le C_\nu \left(\frac{1}{j}\right)^{\nu/2} \quad \forall j$

$(j\lambda)^{\nu/2}\left(1 - \frac{(j\lambda)}{j}\right)^j \le C_\nu$

$\Gamma \; e^\xi \ge 1+\xi$

$e^{-t/j} \ge (1 - t/j)$

$e^{-t} \ge (1 - t/j)^j \; \lrcorner$

$\Rightarrow (j\lambda)^{\nu/2}\left(1 - \frac{(j\lambda)}{j}\right)^j \le (j\lambda)^{\nu/2} e^{-j\lambda}$ bdd $\forall \nu \ge 0$

qualification $\nu_0 = \infty$