Inverse Problems 6.11.2013

1.4.1 The discrepancy principle

$$Tf = g^\delta, \quad T \in K(X,Y)$$

Reconstruction operator $R_\alpha = q_\alpha(T^*T)T^*$
with $\{q_\alpha\}_{\alpha\in A}$ a filter

discrepancy principle: $\alpha_d(\delta, g^\delta) = \sup_{\alpha\in A}\{\|\underbrace{g^\delta - TR_\alpha g^\delta}_{= r_\alpha(TT^*)g^\delta}\|_Y \le \tau\delta\}$

with $\tau > c_r := \sup\{|r_\alpha(\lambda)| : \lambda \in [0, \|T\|^2], \alpha \in A\} \geq 1$

Thm 1.4.1 C: $f \in K_{\nu,\varrho}$, $\{q_\alpha\}_{\alpha\in A}$ has qualification $\nu_0 \geq \nu+1$.
Then $(R_\alpha, \alpha_d(\delta, g^\delta))$ is an order optimal deterministic regularization method

means

$$\|f - R_{\alpha_d} g^\delta\|_X \leq C\varrho^{\frac{1}{\nu+1}}\delta^{\frac{\nu}{\nu+1}}$$

Remark: Thm 1.3.5 G tells that an a priori pcr (parameter choice rule) based on knowledge of ν, ϱ gives order optimality already for $\nu_0 \geq \nu$

Ex: Tikhonov regularization: $\nu_0 = 2$

- $\to$ a priori p.c.r. achieves convergence $O(\delta^{2/3})$
- $\to$ discrepancy principle: cvg $O(\delta^{1/2})$
  - $\to$ "sub-optimal"

There's a remedy by modified discrepancy principle

$$\alpha_d^{mod}(\delta, g^\delta) := \sup_{\alpha\in A}\{\|s_\alpha(TT^*)g^\delta\|_Y \le \tau\delta\}$$

$\uparrow$ special fct specifically chosen to fit q_α (known for Tikhonov)

1.4.2 Discrepancy principle for Tikhonov

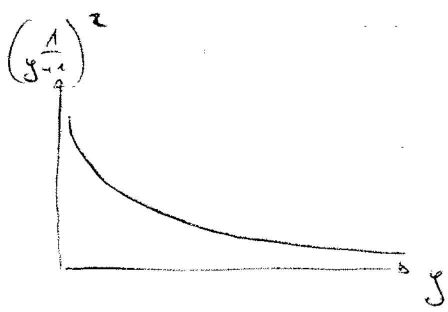
$$R_\alpha := (T^*T + \alpha\,Id)^{-1}T^*, \quad q_\alpha = \frac{1}{\alpha+\lambda} \to r_\alpha = \frac{\alpha}{\alpha+\lambda} \quad : c_r = 1$$

Norm of residual:

$$\|g^\delta - TR_\alpha g^\delta\|_Y^2 = \sum_{j=1}^\infty |r_\alpha(\sigma_j^2)|^2 |(g^\delta, g_j)_Y|^2 =$$

$$= \sum_{j=1}^\infty \left(\frac{\alpha}{\alpha+\sigma_j^2}\right)^2 |(g^\delta, g_j)_Y|^2$$

$\psi(\gamma) = \| g^\delta - TR_{1/\gamma} g^\delta \|^2$



Task: solve $\psi(\gamma) = (\tau\delta)^2$

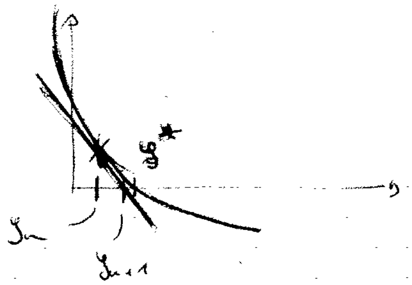
$$\psi(\gamma) = \sum_{j=1}^{\infty} \underbrace{\left(\frac{1}{\gamma\sigma_j^2+1}\right)^2}_{\text{decreasing, convex}} |(g^\delta, g_j)|^2$$

$\Rightarrow$ ψ is decreasing, convex, $\to 0$ for $\gamma \to \infty$

$\psi(0) = \|g^\delta\|_Y^2 > (\tau\delta)^2$: sufficiently large signal-to-noise ratio

$\Rightarrow$ (*) has a unique solution

Recall: Newton's method for $F(\gamma) = 0$: $\gamma_{n+1} = \gamma_n - F(\gamma_n)/F'(\gamma_n)$



F decreasing and convex

if $\gamma_n < \gamma^*$ ($F(\gamma^*) = 0$) (for convex fcts the tangent is allways below the function)

$\Rightarrow$ $\gamma_n < \gamma_{n+1} < \gamma^*$

$\Rightarrow$ monotone convergence of Newtons method

Here: $F(\gamma) = \psi(\gamma) - (\tau\delta)^2$

Introduce $\varphi(\alpha) = \|g^\delta - TR_\alpha g^\delta\|_Y^2$

$= (g^\delta - TR_\alpha g^\delta, g^\delta - TR_\alpha g^\delta)_Y$

$$\frac{d\varphi}{\partial\alpha}(\alpha) \overset{\text{product rule}}{=} -2\left(g^\delta - TR_\alpha g^\delta, \frac{d}{d\alpha}(TR_\alpha g^\delta)\right)_Y$$

$$\overset{\text{linearity}}{=} -2\left(g^\delta - TR_\alpha g^\delta, T\underbrace{\left(\frac{d}{d\alpha}R_\alpha g^\delta\right)}_{}\right)_Y$$

by implicit differentiation of

$(T^*T + \alpha Id) R_\alpha g^\delta = T^* g^\delta$

$\Rightarrow \frac{d}{d\alpha}$: $R_\alpha g^\delta + (T^*T + \alpha Id)\frac{d}{d\alpha}R_\alpha g^\delta = 0$

$\Rightarrow$ $F'(\gamma) = -\frac{1}{\gamma^2}\varphi'\left(\frac{1}{\gamma}\right)$

Newton method for discrepancy principle (initial guess $\gamma_0 > 0$)

Repeat

Solve $(T^*T + \frac{1}{\gamma_k}Id)f_k = T^*g^\delta$

Solve $(T^*T + \frac{1}{\gamma_k}Id)d_k = -f_k$

...

Newton method for discrepancy principle ($\gamma_0 > 0$)

REPEAT
$$\left[\begin{array}{l} \text{Solve} \quad (T^*T + \frac{1}{\gamma_k} Id) f_k = T^* g^\delta \\ \text{Solve} \quad (T^*T + \frac{1}{\gamma_k} Id) d_k = -f_k \\ u_k = g^\delta - T f_k \quad \text{(residual)} \\ \kappa = -2 (u_k, T d_k)_Y \\ \gamma_{k+1} = \gamma_k + \underbrace{\frac{\|u_k\|_Y^2 - (\tau\delta)^2}{\kappa} \gamma_k^2}_{\Delta \gamma_k \; (>0 \text{ for } \gamma_k \text{ small enough})} \end{array}\right.$$

UNTIL $(|\Delta \gamma_k| \le tol \cdot |\gamma_k|)$

1.4.3 Iterative regularization revisited

Recall: Landweber iteration ($0 < \mu < 1/\|T\|^2$) damping parameter

$$\hat{f}_0 := 0, \quad \hat{f}_{n+1} = \hat{f}_n - \mu T^*(g^\delta - T\hat{f}_n)$$

$$R_n g^\delta := \hat{f}_n = \text{filtering reconstruction}$$

$$\left[\quad q_n(\lambda) = \sum_{j=0}^{n-1} (1-\mu\lambda)^j \rightsquigarrow r_n(\lambda) = (1-\mu\lambda)^n \right]$$

q_n and r_n are polynomials

qualification $\nu_0 = \infty$

$$g^\delta - T\hat{f}_n = r_n(TT^*) g^\delta$$

$r_n \in \mathcal{P}_n$ (polynomial of degree n), $r_n(0) = 1$

Idea: define reconstruction method by specifying suitable $r_n \in \mathcal{P}_n$, $r_n(0) = 1$

$\Rightarrow$ "Optimal" choice

$$r_n = \operatorname{argmin} \{ \|r_n(TT^*) g^\delta\|_Y : r_n \in \mathcal{P}_n, r_n(0) = 1 \}$$

$$\begin{cases} r_n(\lambda) = 1 - \lambda q_n(\lambda), & q_n(\lambda) \in \mathcal{P}_{n-1} \\ q_n(\lambda) = \sum_{j=0}^{n-1} \alpha_j \lambda^j \end{cases}$$

$$\Rightarrow \underbrace{r_n(TT^*) g^\delta}_{\text{residual}} = g^\delta - TT^* \sum_{j=0}^{n-1} \alpha_j (TT^*)^j g^\delta$$

$$\Rightarrow \underbrace{r_n(TT^*) g^\delta}_{= g^\delta - T\hat{f}_n} \in g^\delta + TT^* \underbrace{\operatorname{span}\overbrace{\{ g^\delta, TT^* g^\delta, (TT^*)^2 g^\delta, \dots, (TT^*)^{n-1} g^\delta \}}^{n \text{ vectors}}}_{\text{Krilov space } K_n(TT^*, g^\delta)}$$