Motivates $\hat{f}_n \in \text{span}\{T^*g^\delta, (T^*T)T^*g^\delta, \dots, (T^*T)^{n-1}T^*g^\delta\}$

$$= K_n(T^*T, T^*g^\delta) \subset N(T)^\perp$$

r_n = argmin $\Rightarrow$ $\boxed{\hat{f}_n = \underset{f}{\text{argmin}}\{\|g^\delta - Tf\| : f \in K_n(T^*T, T^*g^\delta)\}}$ $\Rightarrow$ defines a method (unique solution) (1.4.3.c)

The method is the conjugate gradient

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Note $K(T^*T, T^*\delta) = T^* K(TT^*, g^\delta)$

(1.4.3.c) with $\hat{f}_n = T^*z$, $z \in Y$

z is minimizer of $\|g^\delta - TT^*z\|_Y$ over $K_n(TT^*, g^\delta)$

$\Rightarrow$ $(\underbrace{g^\delta - TT^*z}, TT^*w)_Y = 0 \quad \forall w \in K(TT^*, g^\delta)$

= residual $g^\delta - T\hat{f}_n =: \rho_n$

$\Rightarrow$ $(\rho_n, TT^*w)_Y = 0 \quad \forall w \in K_n(TT^*, g^\delta)$

$\rho_k = g^\delta - \underbrace{T\hat{f}_k}_{\in K_{k+1}(TT^*, g^\delta)}$, $\quad f_k \in T^* K_k(TT^*, g^\delta)$

If $k<n \Rightarrow (T^*\rho_n, T^*\rho_k)_X = 0$, TT^*-orthogonality of residuals

.... [exploit orthogonality and Gram-Schmidt orthogonalization] + introduction of auxiliary vectors

$\Rightarrow$ Three-term recurrence: cf. "matrix CG" for $T^*Tf = T^*g^\delta$

Initial guess: $\hat{f}_0 := 0$, $\rho_0 = g^\delta$

$p_1 = d_1 := T^*\rho_0$, $n=1$

WHILE ($d_n \neq 0$) (termination)

$$\begin{aligned}
&\eta_n = Tp_n \\
&\alpha = \|d_{n-1}\|_X^2 / \|d_n\|_X^2 \\
&\hat{f}_n = \hat{f}_{n-1} + \alpha p_n \\
&\rho_n = \rho_{n-1} - \alpha \eta_n \quad \text{(residual)} \\
&d_n = T^*\rho_n \\
&\beta = \|d_n\|_X^2 / \|d_{n-1}\|_X^2 \\
&p_{n+1} = d_n + \beta p_n
\end{aligned}$$

- If $d_n = 0 \iff T^* s_n = T^*(g^\delta - T\hat{f}_n) = 0 \Rightarrow g^\delta \in D(T^+)$

$\rightarrow$ $\hat{f}_n$ is a least squares solution and also $\hat{f}_n \in N(T)^\perp$ $\Big\} \Rightarrow \hat{f}_n = T^+ g^\delta$

If g^δ is not in $D(T^+) \Rightarrow$ we expect $\|\hat{f}_n\|_X \to \infty$ as $n \to \infty$
(CG will never stop)

- $(1.4.4.c) \Rightarrow \hat{f}_n = q_n(T^*T)T^*g^\delta$
with some $q_n \in P_{n-1}$ that depends on g^δ

In fact it is $q_n(g^\delta, T^*T)$!

$\Rightarrow$ R_n from CG is <u>non-linear</u> !

<u>Thm 1.4.3.E:</u> $g^\delta \in \overline{R(T)} \Rightarrow \|g^\delta - T\hat{f}_n\|_Y \to 0$ as $n \to \infty$

Proof: $\|g^\delta - T\hat{f}_n\|_Y^2 = \sum_j r_n(g^\delta; \sigma_j^2)^2 \, |(g^\delta, g_j)_Y|^2$

$$\le \sum_j \hat{r}_n(\sigma_j^2)^2 |(g^\delta, g_j)|^2 \quad \forall\, \hat{r}_n \in P_{n+1},\ \hat{r}_n(0) = 1$$

$\varepsilon > 0 : N_\varepsilon \in \mathbb{N} \quad \sum_{j > N_\varepsilon} |(g^\delta, g_j)_Y|^2 \le \varepsilon$

By density of polynomials (Stone-Weierstrasse's thm)
Pick $\varphi \in C^0([0, \|T\|^2])$, $\|\varphi\|_\infty \le 1$

$\varphi(0) = 1$, $\varphi(\lambda) = 0$ if $\lambda \geq \sigma_{N_\varepsilon}^2$

$\exists\ \hat{p} \in P_n : \hat{p}(0) = 1 \qquad \|\hat{p} - \varphi\|_{L^\infty(0, \|T\|^2)} \le \varepsilon$

$$\sum_j \hat{p}(\sigma_j^2) \, |(g^\delta, g_j)_Y|^2 \le \sum_{j > N_\varepsilon} \underbrace{\hat{p}(\sigma_j^2)}_{\le 1+\varepsilon} |(g^\delta, g_j)_Y|^2 + \sum_{j \le N_\varepsilon} \underbrace{|\hat{p}(\sigma_j)^2|}_{\le \varepsilon} |(g^\delta, g_j)|$$

$$\le (1+\varepsilon)\varepsilon + \varepsilon \|g^\delta\|$$

<u>1.4.4 Stopping rules</u>

Thm 1.3.5 G: An a priori stopping rule for Landweber iteration that yields an order optimal deterministic regularization method

CG (Plato): No regularization method for any a priori stopping rule

Stopping rule by discrepancy principle

STOP, if $\| g^\delta - T\hat{f}_n \|_Y \le \tau\delta$, $\tau > 1$ (1.4.4.A)

for the first time

[Assume: $\| g^\delta \|_Y > \tau\delta$, $\hat{f}_0 = 0$]

Will we stop?

CG: Thm 1.4.3.E : ✓

Landweber: ($\mu < 1/\|T\|^2$) $\| g^\delta - T\hat{f}_n \|_Y \le \overbrace{\| (Id - \mu TT^*)^n g \|_Y}^{\to 0}$
$+ \underbrace{\| \underbrace{(Id - \mu TT^*)^n}_{\| \; \| \le 1} (g - g^\delta) \|_Y}_{\le \delta}$

<u>Thm 1.4.4.B</u> for Landweber assume

$\mu < 1/\|T\|^2$, $g = Tf^+$

$\| g^\delta - T\hat{f}_n \|_Y > 2\delta \Rightarrow \| f^+ - \hat{f}_{n+1} \|_X \le \| f^+ - \hat{f}_n \|_X$

$\Rightarrow$ Choose $\tau = 2$, and with Landweber you improve at any step

For Landweber Bound on $n = n(\delta, g^\delta)$
↑
from discrepancy principle (1.4.4.A)

$$\| g^\delta - T\hat{f}_n \|_Y \le \underbrace{\| (Id - \mu TT^*)^n g \|_Y}_{\text{should be } \le (\tau - 1)\delta} + \underbrace{\| (Id - \mu TT^*)(g - g^\delta) \|_Y}_{\le \delta} \overset{!}{\le} \tau\delta$$

$$\| (Id - \mu TT^*)^n g \|_Y^2 \overset{SVD}{=} \sum_j (1 - \mu\sigma_j^2)^{2n} |(g - g_j)_Y|^2$$

$$= \sum_j \sigma_j^2 (1 - \mu\sigma_j^2)^{2n} |(f^+, f_j)_X|^2$$

Assumption: g attainable: $g = Tf^+$

A priori knowledge: $f^+ \in \mathcal{M}_{\nu, S}$, $\nu \ge 0$

$$= \sum_j (\mu\sigma_j^2)^{(1+\nu)} (1 - \mu\sigma_j^2)^{2n} \sigma_j^{-2\nu} |(f^+, f_j)|^2 \cdot \frac{1}{\mu^{1+\nu}}$$

$\ulcorner$ $\varphi(y) = y^{1+\nu}(1 - y)^{2n}$, $0 \le y \le 1$

$$\varphi(y) \le \varphi\Big(\underbrace{\frac{1+\nu}{2n+1+\nu}}_{\text{maximizer}}\Big) = \left(\frac{1+\nu}{2n+1+\nu}\right)^{(1+\nu)} \underbrace{\left(1 - \frac{1+\nu}{2n+1+\nu}\right)^{2n+1+\nu}}_{\le e^{-(1+\nu)}} \left(\frac{2n+1+\nu}{2n}\right)^{1+\nu}$$

$$\le C \left(\frac{1}{2n}\right)^{\nu+1}$$
$\llcorner$

$$\| (Id - \mu TT^*)^n g \|_Y \leq C \left(\frac{1}{2n}\right)^{\nu+1} \|f\|_\nu^2 \frac{1}{\mu^{\nu+1}} \doteq (\tau-1)^2 \delta^2$$

$$\Rightarrow \boxed{n \sim \delta^{-2/(\nu+1)} \cdot C}$$

More, since Landweber has infinite qualification (1.4.4.A) will yield an order optimal regularization method (→ 1.4.2)