$$\| (Id - \mu TT^*)^n g \|_Y \le C \left(\frac{1}{2n}\right)^{\nu+1} \|f\|_V^2 \frac{1}{\mu^{\nu+1}} \doteq (\tau-1)^2 \delta^2$$

$$\Rightarrow \boxed{n \sim \delta^{-2/(\nu+1)} \cdot C}$$

More, since Landweber has infinite qualification (1.4.4.A) will yield an order optimal regularization method ($\to$ 1.4.2)

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STOP iteration, when $\| g^\delta - TR_n g^\delta \|_Y \le \tau \delta$

Landweber: if $f^+ \in K_{\nu,\varrho}$ then
- $\| f^+ - R_{n_{stop}} g^\delta \|_X \le C \delta^{\frac{\nu}{\nu+1}} \varrho^{\frac{1}{\nu+1}}$
- $n_{stop} \approx \delta^{-2/\nu+1}$

order optimality

CG: [Plato 1992]: If $g \in R(T)$, $f^+ \in K_{\nu,\varrho}$
$(\hat{f}_n) \mathrel{\hat{=}}$ CG sequence with $\hat{f}_0 = 0$,
$n_{STOP} = n_{STOP}(\delta, g^\delta)$ by discrepancy principle
$\| f^+ - \hat{f}_{n_{STOP}} \|_X \le C \delta^{\frac{\nu}{\nu+1}} \varrho^{1/\nu+1}$

[Nemirovski 1986]: $n_{STOP} \approx \delta^{-\frac{1}{\nu+1}}$ $\left(< \delta^{-2/\nu+1} \text{ for Landweber} \right)$

1.5 Linear Stochastic Inverse Problems

1.5.1 Estimation of probability density

Estimation is the equivalent of reconstruction

Related to differentiation

Setting: $(\Omega, \mathcal{A}, \mathbb{P}) \mathrel{\hat{=}}$ probability space
Random variable (RV) $X: \Omega \to [0,1]$
X has (probability) density $f \in L^1(\mathbb{R})$
$\mathbb{P}(X \in [a,b]) = \int_a^b f(t)\,dt$

Cumulative distribution function (CDF) F
$$F(x) = \mathbb{P}(X \le x) = \int_0^x f(t)\,dt =: (Tf)(x)$$

Given: sample of n iid copies of X:
RV $X_1, \dots, X_n$

Sought: Estimate for $f \in L^2(0,1)$

Estimate for CDF: $\hat{F}(x,\omega) = \frac{1}{n} \# \{ i \; X_i(\omega) \le x \}$

this is a random variable
a step function (for ω fixed)

$$\hat{F}(x,\omega) = \frac{1}{n}\sum_{i=1}^{n} \chi_{\{X_i(\omega)\le x\}}(x)$$

$\uparrow$ characteristic function

Expectation: $\mathbb{E}\left(\hat{F}(x,\omega)\right) = \frac{1}{n}\sum_{i=1}^{n} \underbrace{\mathbb{E}\left(\chi_{\{X_i\le x\}}(x)\right)}_{} = F(x)$

$$= \int_{\{\omega: X_i(\omega)\le x\}} 1 \, dP = \mathbb{P}(X_i \le x) = F(x)$$

$\longrightarrow$ $\hat{F}$ unbiased estimator

$$\mathrm{Var}\left(\hat{F}(x,\omega)\right) = \frac{1}{n^2}\mathrm{Var}\left(\sum_i \chi_{\{\dots\}}\right) = \frac{1}{n^2}\sum_i \mathrm{Var}\left(\chi_{\{X_i(\omega)\le x\}}\right)$$

$\chi_{\{\}}$ independent

$$= \frac{1}{n^2}\sum_i \mathbb{E}\left(\chi^2_{\{X_i(\omega)\le x\}}\right) - \mathbb{E}\left(\chi_{\{..\}}\right)^2$$

$$= \frac{1}{n^2}\sum_i \left(F(x) - F(x)^2\right)$$

$$= \frac{1}{n}\underbrace{\left(F(x) - F(x)^2\right)}_{\le 1/4}$$

Recovery of f through difference quotient

$$\hat{f}_{\Delta t}(x,\omega) = \frac{\hat{F}(x+\Delta t,\omega) - \hat{F}(x-\Delta t,\omega)}{2\Delta t} =: R_{\Delta t}\hat{F}$$

$$= \frac{1}{2\Delta t\cdot n}\#\left\{x - \Delta t \le X_i(\omega) \le x + \Delta t\right\}$$ "Histogram estimator"

$$\mathbb{E}\left(\hat{f}_{\Delta t}(x,\omega)\right) = \frac{1}{2\Delta t\cdot n}\sum_{i=1}^{n}\mathbb{P}\left(x-\Delta t \le X_i(\omega) \le x+\Delta t\right)$$

$$= \frac{1}{2\Delta t}\left(F(x+\Delta t) - F(x-\Delta t)\right) = R_{\Delta t}F$$

Wrap up. Perturbed ill-posed operator equation

$$Tf = \underset{\substack{\uparrow\\ \text{exact data}}}{F} + \underbrace{(\hat{F} - F)}_{\text{RV with values in } L^2(0,1),\ \mathbb{E}(\ )=0}$$

Noise level $\delta \sim \underbrace{\sqrt{\mathrm{Var}\left(\|\hat{F}-F\|_Y^2\right)}}_{\approx \frac{1}{\sqrt{n}}}$

Gauging the estimation error $f(x) - \hat{f}_{\Delta t}(x,\omega)$

Mean integrated square error (MISE)

$$\mathbb{E}\left(\|f - \hat{f}_{\Delta t}\|^2_{L^2}\right) = \mathbb{E}\left(\|f - R_{\Delta t}F + R_{\Delta t}F - \hat{f}_{\Delta t}\|^2_{L^2}\right)$$

$$= \underbrace{\mathbb{E}\left(\|f - R_{\Delta t}F\|^2_{L^2}\right)}_{= \|f - R_{\Delta t}F\|^2_{L^2}} + 2\,\underbrace{\mathbb{E}\left(\left(f - R_{\Delta t}F,\ R_{\Delta t}F - \hat{f}_{\Delta t}\right)_{L^2}\right)}_{\overset{\text{Fubini}}{=} \left(f - R_{\Delta t}F,\ \underbrace{\mathbb{E}\left(R_{\Delta t}F - \hat{f}_{\Delta t}\right)}_{=0}\right)_{L^2} = 0} + \underbrace{\mathbb{E}\left(\|R_{\Delta t}F - R_{\Delta t}\hat{F}\|^2_{L^2}\right)}_{\text{Variance}}$$

$\|f - R_{\Delta t}F\|^2_{L^2}$: approximation error (or bias)

$= \mathbb{E}(R_{\Delta t}\hat{F})$

<u>Examine Variance</u>

$$\int_\Omega \int_0^1 |R_h F(x) - \hat{f}_{\Delta t}(x,\omega)|^2 \, dx \, d\mathbb{P}(\omega)$$

$$\int_0^1 \int_\Omega |(\mathbb{E}\hat{f}_{\Delta t})(x) - \hat{f}_{\Delta t}(x,\omega)|^2 \, d\mathbb{P}(\omega)\, dx$$

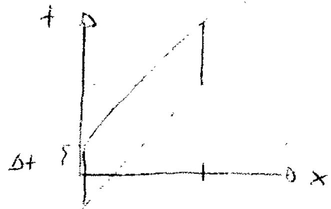
$$= \int_0^1 \operatorname{Var} \hat{f}_{\Delta t}(x,\omega)\, dx =$$

$$\operatorname{Var}(\hat{f}_{\Delta t}(x,\omega)) \underset{\text{wc independence}}{=} \frac{1}{4\Delta t^2 n^2} \sum_{i=1}^n \operatorname{Var}\left(\chi_{\{x-\Delta t \le X_i(\omega) \le x+\Delta t\}}\right)$$

$$= \frac{1}{4\Delta t^2 n} \operatorname{Var}(\chi_{\{\dots\}}) = \frac{1}{4\Delta t^2 n}\left(\mathbb{E}(\chi^2_{\{\dots\}}) - \mathbb{E}(\chi_{\{\dots\}})^2\right) =$$

$$= \frac{1}{4\Delta t^2 n}\left(\mathbb{P}(x-\Delta t \le X \le x+\Delta t) - \mathbb{P}(x-\Delta t \le X \le x+\Delta t)^2\right)$$

$$\le \frac{1}{4\Delta t^2 n} \int_{x-\Delta t}^{x+\Delta t} f(t)\, dt$$



$$\le \int_0^1 \frac{1}{4\Delta t^2 n} \int_{x-\Delta t}^{x+\Delta t} f(t)\, dt \, dx$$

$$\le \frac{1}{4\Delta t^2 n} \int_{-\Delta t}^{1+\Delta t} f(t)\, dt \cdot 2\Delta t \le \frac{1}{2\Delta t\, n} \underbrace{\|f\|_{L^1(0,1)}}_{=1} = \frac{1}{2\Delta t\, n}$$