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$\hat{f}_{\Delta t}(x,\omega) = R_{\Delta t}\hat{F}(x,\omega)$

$F(x) = \int_0^x f(t)\,dt = Tf(x)$

MISE: $\mathbb{E}\left( \| \hat{f}_{\Delta t}(x,\omega) - f(x,\omega) \|^2_{L^2(0,1)} \right) =$

$= \int_\Omega \int_0^1 | f_{\Delta t}(x,\omega) - f(x,\omega)|^2 \,dx\, d\omega$

$= \underbrace{\mathbb{E}\left( \| \hat{f}_{\Delta t} - R_{\Delta t} F \|^2_{L^2} \right)}_{= Var(\| f_{\Delta t} - R_{\Delta t}F\|_{L^2})} + \underbrace{\| R_h F - f \|^2_{L}}_{\text{bias}}$

Variance

Bound for variance $\mathbb{E}\left( \| \hat{f}_{\Delta t} - R_{\Delta t} F \|^2_{L^2} \right) \le \frac{1}{2 \Delta t\, n}$

Bound for bias: a priori knowledge about the smoothness of f is essential

By Taylor's formula:

$f' \in L^2 \to \| (R_{\Delta t} T - Id) f \|_{L^2} \le \frac{1}{2} \Delta t \| f' \|_{L^2}$ (a)

$f'' \in L^2 \to \| (R_{\Delta t} T - Id) f \|_{L^2} \le 1/6 \, \Delta t^2 \| f'' \|_{L^2}$ (b)

A priori knowledge (source condition)

(a) $MISE \le \frac{1}{2\Delta t\, n} + \frac{1}{4} \Delta t^2 \| f' \|^2_{L^2} \longrightarrow \min$ wrt $\Delta t$

$\Delta t_* = \left( \frac{1}{n \| f' \|_{L^2}} \right)^{1/3}$ : $MISE = O(n^{-2/3})$

(b) $\Delta t_* = O(n^{-1/5}) \Rightarrow MISE = O(n^{-4/5})$

Here: Standard deviation $O\left(\frac{1}{\sqrt{n}}\right)$

"Stocastic noise level" $\delta = \frac{1}{\sqrt{n}}$

(a) $\sqrt{MISE} \le O(\delta^{2/3})$

(b) $\sqrt{MISE} \le O(\delta^{4/5})$

In deterministic setting

$$\| R_{\Delta t} g^\delta - f \|_{L^2} \le \begin{cases} O(\delta^{1/2}) & \text{in (a)} \\ O(\delta^{2/3}) & \text{in (b)} \end{cases}$$

Remark: $\mathbb{E}(\|\hat{F}-F\|_{L^2}^2) = \mathcal{O}(1/n)$

$\sqrt{\text{MISE}(\hat{F}-F)} = \mathcal{O}(\delta)$

this is not an inverse problem because we get an optimal rate in the noise level, i.e., for cumulative fct,

## 15.2 General stochastic noise models

$X, Y$ Hilbert spaces, $(\Omega, \mathbb{P}, \mathcal{A})$ probability space

$$Tf = g + \underset{\substack{\uparrow \\ \text{RV with values in } Y,\ Z: \Omega \to Y}}{Z} \qquad \mathbb{E}(Z) = \int_\Omega Z(\omega)\, d\mathbb{P}(\omega) \in Y$$

Assume $\mathbb{E}(\|Z\|_Y^2) < \infty$

$\|Z\|^2 := \mathbb{E}(\|Z\|_Y^2)$

Covariance: $\text{Cov}(Z) \in \mathcal{L}(Y \times Y, \mathbb{R})$

$$\text{Cov}(Z)(g_1, g_2) := \text{Cov}((Z, g_1)_Y, (Z, g_2)_Y)$$

$$\text{Cov}(X_1, X_2) = \int_\Omega (X_1(\omega) - \mathbb{E}[X_1])(X_2(\omega) - \mathbb{E}[X_2])\, d\mathbb{P}(\omega)$$

$$\text{Var}(Z) = \int_\Omega \|Z(\omega) - \mathbb{E}[Z]\|_Y^2\, d\mathbb{P}(\omega)$$

$\text{Cov}(Z)$ is symmetric

$$\|\text{Cov}(Z)\| = \sup_{\|g_1\|, \|g_2\| \le 1} \text{Cov}(Z)(g_1, g_2) = \sup_{\|g\| \le 1} \text{Cov}(Z)(g, g)$$

$$= \text{Var}(Z)$$

$$= \sup_{\|g\| \le 1} \int_\Omega (Z(\omega), g)_Y^2\, d\omega \le \text{Var}(Z)$$

(maybe equality? or maybe not)

Rmk. another interpretation of covariance,

since $\mathcal{L}(Y \times Y, \mathbb{R}) = \mathcal{L}(Y, Y)$ (isomorphismus)

$\to \text{Cov}(Z): Y \to Y$

Abuse of notation

$$(\text{Cov}(Z) g_1, g_2)_Y := \text{Cov}(Z)(g_1, g_2)$$

**Thm 15.2.A** (Bias-Variance Decomposition)

If $\mathbb{E}(Z) = 0$, $R \in \mathcal{L}(Y, X)$ [estimator], $Tf^+ = g$

$$\mathbb{E}(\|R(g+Z) - f^+\|_X^2) = \underbrace{\|(RT - \text{Id}) f^+\|_X^2}_{\substack{\text{bias} \\ \Updownarrow \\ \text{approximation}}} + \underbrace{\mathbb{E}(\|RZ\|_X^2)}_{\substack{\text{variance} \\ \Updownarrow \\ \text{data noise error}}}$$

Assuming $Z$ to be a $Y$-valued RV with finite Variance is too restrictive.

Example: $\{g_j\}$ ONB of $Y$

$$Z(\omega) = \sum_j \underbrace{X_j(\omega)}_{iid} g_j \qquad (*)$$

$$g(x) - \hat{g}(x,\omega) = \delta \sum_{j=1}^{\infty} \underbrace{Y_j(\omega)}_{\text{iid uniform in } [-1,1]} \sin(\pi j x)$$

→ White noise model (the sequence may not converge)

$$(*) \Rightarrow \int_\Omega \| Z(\omega) \|_Y^2 \, d\mathbb{P}(\omega) = \sum_j \mathbb{E}(X_j^2) = \infty$$

Def 15.2.3 a Hilbert space process (HSP) is a bdd linear mapping $Y \to L^2(\Omega, \mathbb{P})^{(*)}$

(*) space of $\mathbb{R}$-RV with finite variance

If $\hat{Z}$ is a $Y$-valued RV $\Rightarrow$ $g \to \{\omega \to (\hat{Z}(\omega), g)_Y\}$ is HSP

$$\int_\Omega |(\hat{Z}(\omega), g)_Y|^2 \, d\mathbb{P}(\omega) \le \|g\|^2 \int_\Omega \| \hat{Z}(\omega) \|_Y^2 \, d\mathbb{P}(\omega)$$

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Repetition: $Tf = g + Z$, $T \in K(X,Y)$
[Assume: $g = Tf^+$, $T$ injective]

First $Z \mathrel{\hat{=}} Y$-valued RV

$$\int_\Omega \| Z(\omega) \|^2 \, d\mathbb{P}(\omega) < \infty$$

"Variance": $Var(Z) = \int_\Omega \| Z(\omega) - \mathbb{E}(Z) \|_Y^2 \, d\mathbb{P}(\omega)$

$$Cov(Z)(g_1, g_2) = \int_\Omega (Z(\omega) - \mathbb{E}(Z), g_1)_Y \, (Z(\omega) - \mathbb{E}(Z), g_2)_Y \, d\mathbb{P}\omega$$

$\Rightarrow \| Cov(Z) \| \le Var(Z)$

Note: $Cov(Z)$ symmetric, non-negative

"Variance is a trace"

$\{e_j\} \hat{=}$ ONB of $Y$, assume $\mathbb{E}(Z) = 0$

$$\sum_j (Cov(Z)e_j, e_j)_Y = \int_\Omega \sum_j (Z(\omega), e_j)^2 \, dP(\omega)$$

monotone convergence thm

$$= \int_\Omega \|Z(\omega)\|^2 \, dP(\omega) = Var(Z)$$

Bessel inequality

<u>Def</u>: $S \in \mathcal{L}(Y)$, trace $tr(S) := \sum_j (Se_j, e_j)_Y$ for an ONB $\{e_j\}$ of $Y$

⌜Well-defined, if finite? Another ONB $\{e_j'\}$ $\quad [(x,y)_Y = \sum_j (x,e_j)_Y (y, e_j)_Y]$

$$\sum_j (S e_j', e_j')_Y = \sum_{j,\ell} (S e_j', e_\ell)_Y (e_j', e_\ell)_Y =$$

$$= \sum_\ell \sum_j (e_j', S^* e_\ell)_Y (e_j', e_\ell)_Y = \sum_\ell (e_\ell, S^* e_\ell) = \sum_\ell (S e_\ell, e_\ell)$$

⌞

<u>Def</u>: $S \in K(Y)$ is nuclear (or trace class) if

$\sum_j \sigma_j(S) < \infty$, (the sum of its singular values should be finite)

where $(\sigma_j(S))_j$ are the singular values of $S$

⌜ Notation: $\mathcal{L}_1(Y)$ is the set of trace class op.

$((\sigma_j)_j \in \ell^1(\mathbb{N}))$

because $(\sigma_j, u_j, v_j) \hat{=}$ SVD system of $S$

⌞ $\to tr(S) = \sum_j (Su_j, u_j)_Y = \sum_j \sigma_j (v_j, u_j)_Y \overset{CS}{\le} \sum \sigma_j$

Example: White noise has not bdd variance

$Y = L^2(0,1)$

"$Z(\omega)(x) = \sum_j X_j(\omega) \sin(j\pi x)$"

iid with values in $[0,1]$