Def 1.5.2.B: Hilbert space process (HSP) is a bdd linear mapping $Y \to L^2(\Omega, \mathbb{R})$

If $\hat{z}$ is Y-valued RV, bdd variance

$\Rightarrow$ $z : Y \to L^2(\Omega, \mathbb{R})$ is a HSP with
$$g \mapsto (\omega \mapsto (\hat{z}(\omega), g)_Y)$$

$$\underset{\underset{\|g\|_Y \le 1}{\sup_{g \in Y}} \|z(g)\|_{L^2(\Omega,\mathbb{R})}}{\underset{\uparrow}{\|z\|}} \le \int_\Omega \|\hat{z}(\omega)\|_Y^2 \, dP\omega$$

White noise (generalized)

$\{g_j\}$ ONB of Y,

$$z_w(g)(\omega) := \sum_j \underset{\substack{\uparrow \\ \text{iid, finite range } (-1,1),\ \mathbb{E}(X_j) = 0}}{X_j(\omega)} (g, g_j)_Y$$

$$\int_\Omega |z_w(g)(\omega)|^2 \, dP(\omega) = \operatorname{Var}(z_w(g)) \underset{\text{indep}}{=} \sum_j (g, g_j)_Y^2 \operatorname{Var}(X_j)$$
$$= \operatorname{Var}(X_1) \|g\|_Y^2$$

z HSP: $(\underbrace{\mathbb{E}(z)}_{\in Y}, g)_Y = \mathbb{E}(z(g))$, $g \in Y$

$$\operatorname{Cov}(z)(g_1, g_2) := \operatorname{Cov}(z(g_1), z(g_2))$$
$[= \operatorname{Cov}(\hat{z})$, if z generated by Y-RV $\hat{z}]$

$$\operatorname{Cov}(z_w)(g_1, g_2) = \operatorname{Cov}\Big(\sum_j X_j(\omega)(g_1, g_j)_Y, \sum_\ell X_\ell(\omega)(g_2, g_\ell)_Y\Big)$$
$$= \sum_j \sum_\ell (g_1, g_j)(g_2, g_\ell) \operatorname{Cov}(X_j, X_\ell)$$

independence $\to$ $= \sum_j \operatorname{Cov}(X_j, X_j)(g_1, g_j)_Y (g_2, g_j)_Y$
$$= \operatorname{Var}(X)(g_1, g_2)_Y$$

$\to$ $\underbrace{\operatorname{Cov}(z_w) = \operatorname{Var}(X) \cdot \mathrm{Id}}_{\text{defines a \underline{white noise} HSP}} \notin \mathcal{L}_1(Y)$

"Variance" of a HSP:

$\hat{z}$ Y-RV $\Rightarrow$ HSP z $\quad \left(E(z) = E(\hat{z}) = 0\right)$

$$E\left(\|\hat{z}\|_Y^2\right) = E\left(\sum_j \left(\hat{z}(\omega), g_j\right)_Y^2\right)$$

$$= \sum_j E\left(\left(z(g_j)\right)^2\right)$$

$\mathrm{Var}(X) = E(X^2) - E(X)^2$

$$= \sum_j \mathrm{Var}\left(z(g_j)\right) + \underbrace{E\left(\left(\hat{z}(\omega), g_j\right)_Y\right)^2}_{=0}$$

$$= \sum_j \left(\mathrm{Cov}(z) g_j, g_j\right)_Y = \mathrm{tr}\left(\mathrm{Cov}(z)\right)$$

$\rightarrow$ If $\mathrm{tr}(\mathrm{Cov}(z)) < \infty$, then use it as a norm for HSP

$\rightarrow$ Meaning of $\| \ \|$

<u>Thm:</u> z HSP, $\mathrm{Cov}(z)$ is nuclear
$\hat{z}(\omega) := \sum_j z(g_j) g_j$, where
$(\sigma_j, g_j, g_j) \hat{=}$ SVD of $\mathrm{Cov}(z)$
is a Y-valued RV with bdd variance

Operators acting on HSP

$R \in \mathcal{L}(Y, X)$

$\hat{z}$ Y-RV $\rightarrow$ $(R\hat{z})(\omega) := R\left(\hat{z}(\omega)\right)$

$\updownarrow$ HSP z $\qquad$ $L_D \times$ RV $\updownarrow$ HSP Rz

$$(Rz)(f)(\omega) = \left(R\hat{z}(\omega), f\right)_X$$
$$= \left(\hat{z}(\omega), R^* f\right)_Y = z\left(R^* f\right)(\omega)$$

<u>Def:</u> z HSP $\qquad (Rz)(f)(\omega) := z(R^* f)(\omega)$

$\rightarrow$ $\mathrm{Cov}(Rz)(f_1, f_2) = \mathrm{Cov}(z)(R^* f_1, R^* f_2)$

$\Leftrightarrow$ $\boxed{\mathrm{Cov}(Rz) = R \circ \mathrm{Cov}(z) \circ R^*}$

Back to stochastic inverse problems:

Reconstruction error (risk) for reconstruction operator $R \in \mathcal{L}(Y, X)$, if noise is HSP

$$\| R(g+Z) - f^+ \|^2$$

Here: $\| \ \| \hat{=}$ norm on X-HSP

Z Y-HSP $\Rightarrow g + Z \hat{=}$ Y-HSP

$R(g+Z) \hat{=}$ X-HSP

If $Z \hat{=}$ Y-RV $\Rightarrow \| R(g+Z) - f^+ \|^2 = \mathbb{E}\left(\| R(g+Z(\omega)) - f^+ \|_X^2 \right)$

In order to obtain finite risk (error), we demand that RZ has nuclear covariance

When is $R \operatorname{Cov}(Z) R^*$ nuclear for general $\operatorname{Cov}(Z) \in \mathcal{L}(Y)$?

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$$Tf = g + Z, \quad T \in K(X, Y)$$

Z: Y-RV or Y-HSP

Y-RV: $\int_\Omega \| Z(\omega) \|^2 \, d\mathbb{P}(\omega) = \operatorname{tr}(\operatorname{Cov}(Z))$ "benign noise"

Y-HSP: (white noise: $\operatorname{Cov}(Z) \sim Id$) "nasty noise"

Reconstruction operator $R \in \mathcal{L}(X, Y)$

$\mathbb{E}\left(\| R(g+Z) - f^+ \|_X^2 \right) < \infty$ is desired

Necessary: RZ is a X-RV with finite variance

$$\operatorname{Cov}(RZ) = R \operatorname{Cov}(Z) R^* \overset{!}{\in} \mathcal{L}_1(X)$$

<u>Def:</u> $R \in \mathcal{L}(Y, X)$ is Hilbert-Schmidt ($R \in \mathcal{L}_2(Y, X)$)

$\Leftrightarrow \sum_j \| R e_j \|_X^2 =: \| R \|_{\mathcal{L}_2}^2 < \infty$ for an ONB $\{e_j\}$ of Y

<u>Facts:</u>
- $\mathcal{L}_2(Y, X) \subset K(Y, X)$
- $\| R \|_{\mathcal{L}_2}^2 = \sum_j \sigma_j^2$, where $\{\sigma_j\}$ are the singular values of R

<u>Thm:</u>
- $R \in \mathcal{L}_2(Y, X)$, $S \in \mathcal{L}(Y) \Rightarrow RS \in \mathcal{L}_2(Y, X)$
- $R \in \mathcal{L}_2(Y, X)$, $S \in \mathcal{L}_2(Y) \Rightarrow RS \in \mathcal{L}_1(Y, X)$