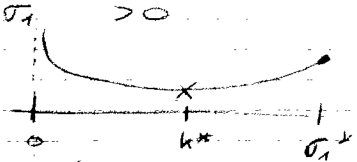
(iii) $k \to 0$: $\varphi(x) \to \infty$ & increasing

$k \to \sigma_1^{\nu}$: $\varphi(\sigma_1^{\nu}) = \rho^2 \sigma_1^{2\nu}$

$$\frac{d\varphi}{dk} = 2\rho^2 k + \varepsilon^2 \sum_{j=1}^{J(k)} 2\sigma_j^{-2-2\nu} \lambda_j^{k}$$

$$\frac{d\varphi}{dk}(\sigma_1^{\nu}) = 2\rho^2 \sigma_1^{\nu} > 0$$



φ convex, differentiable $\quad \Rightarrow \varphi$ is min for $k = k^* \in]0, \sigma_1^{\nu}[$

[$x \mapsto (\max(x,0))^2$ is diffbar]

Thm (5.1.3.H)

minimax linear estimator is $R_{\{\lambda\}}$ with $\lambda_j = \max\{1 - k^*\sigma_j^{-\nu}, 0\}$, $k^* \in]0, \sigma_1^{\nu}[$: $\frac{d\varphi}{dk}(k^*) = 0$ and

$$\text{mlr}(\varepsilon W, K_{\nu,\rho}) = (k^*\rho)^2 + \varepsilon \sum_{j=1}^{\infty} (\lambda_j/\sigma_j)^2$$

But $R_{\{\lambda^{k^*}\}}$ cannot be implemented, because SVD-based

1.5.4 Spectral regulation methods

Example: Spectral cut-off (theoretical tool)

$$R_\alpha^{sc}\, g = \sum_{\sigma_j > \alpha} \sigma_j^{-1} (g, g_j) f_j$$

$$\text{risk}_{R_\alpha^{sc}}(\varepsilon W, K_{\nu,\rho}) = \sigma_{\bar{J}+1}^{2\nu} \rho^2 + \varepsilon^2 \sum_{j=1}^{\bar{J}} \sigma_j^{-2}$$

$$\bar{J} = \max\{j : \sigma_j > \alpha\}$$

Choice $\alpha := \bar{\alpha} = (2k^*)^{1/\nu}$

$\Rightarrow$ for $j \le \bar{J}$: $\lambda_j^{k^*}$ optimal weights from Thm 5.1.3 H

$$\underbrace{\Rightarrow \sigma_{\bar{J}+1}^{2\nu} \le 4(k^*)^2}$$

$$\Rightarrow \text{risk}_{R_{\bar{\alpha}}^{sc}}(\varepsilon W, K_{\nu,\rho}) \le 4(k^*)^2 \rho^2 + \varepsilon^2\, 4 \sum_{j=1}^{\infty} \left(\frac{\lambda_j^{k^*}}{\sigma_j}\right)^2$$

$$= 4\, \text{mlr}(\varepsilon W, K_{\nu,\rho})$$

Filtering estimators: $\{q_\alpha\}_{\alpha \in \mathbb{R}^+} \hat{=}$ filter

$$R_\alpha g = \sum_{j=1}^{\infty} q_\alpha(\sigma_j^2)\,(T^* g, f_j) f_j$$

(F1) $\lim_{\alpha \to 0} q_\alpha(\lambda) = \begin{cases} 1, & \lambda = 0 \\ 0, & \lambda \neq 0 \end{cases}$

(F2) $|r_\alpha(\lambda)| \leq C_r$, $\quad r_\alpha(\lambda) = 1 - \lambda q_\alpha(\lambda)$

(F3a) $\alpha |q_\alpha(\lambda)| \leq C_q^1$, $\quad$ (F3b) $\lambda |q_\alpha(\lambda)| \leq C_q^2$

[(F3a) $\wedge$ (F3b) more restrictive than (F3) in Def 1.3.5.1

Noise model: $Tf = g + \varepsilon Z$

$Z \hat{=} Y$-HSP, $\mathbb{E}(Z) = 0$, $\|Cov(Z)\| \leq 1$

$$\mathbb{E}(\|R_\alpha(Tf^+ + \varepsilon Z) - f^+\|_X^2) = \underbrace{\|(R_\alpha T - Id) f^+\|_X^2}_{\overset{[\text{Thm } 1.3.5G]}{\leq} (C_\nu \alpha^\nu g_1)^2} + \varepsilon^2\, \mathbb{E}(\|R_\alpha Z\|_X^2)$$

if $f \in K_{\nu,g}$ and $\{q_\alpha\}$ has qualification $\nu_1 \geq \nu$

i.e. biggest ν for which $\sup_{0 \leq \lambda \leq \|T\|^2} \lambda^{\nu/2} |r_\alpha(\lambda)| \leq C_\nu \alpha^{\nu/2}$ holds true

$$\mathbb{E}(\|R_\alpha Z\|) = tr\,(Cov(R_\alpha Z)) = tr\,(R_\alpha\, Cov(Z)\, R_\alpha^*) =$$

$$= tr\,(q_\alpha(T^*T)\, T^*\, Cov(Z)\, T\, q_\alpha(TT^*)) =$$

use $\{g_j\}$-ONB

$$= \sum_{j=1}^{\infty} q_\alpha(\sigma_j^2)\, \sigma_j^2\, \underbrace{Cov(Z)(g_j, g_j)}_{\leq 1}$$

$$\Rightarrow \mathbb{E}(\|R_\alpha(Tf^+ + \varepsilon Z) - f^+\|_X^2) \leq (C_\nu \alpha^\nu g)^2 + \sum_{j=1}^{\infty} q_\alpha(\sigma_j)^2\, \sigma_j^2$$

"Nice noise": $Cov(Z)$ nuclear

Variance $\underset{(F3b),(F3a)}{\leq} (C_q^2)\, C_q^1\, \frac{1}{\alpha}\, \|Cov(Z)\|_{\mathcal{L}_1(Y)}$

$\to$ same a priori parameter choice rule as in 1.3

"Nasty noise": limits on ill-posedness

Assumption: $\sigma_j \approx j^{-b/2}$, $b > 1$