(i) $(f_n)_n$ is bounded, $\|u\|_{H^2(0,1)} \le C \|s\|_{L^2(0,1)}$
solution

$\Rightarrow \|\Phi(f_n)\|_{H^2}$ is bdd $\Rightarrow \exists$ H^2-weakly cvg. subsequence $(\Phi(f_n))$

(ii) Rellich thm: $H^2(0,1) \overset{c}{\hookrightarrow} H^1(0,1)$

⌈ Thm. $T \in K(H_1, H_2)$, $x_n \rightharpoonup x$ in H_1 $\Rightarrow$ $T(x_n) \to T(x)$ in H_2 ⌋

$\Rightarrow \Phi(f_n) \to \bar{u}$ in $H_0^1(0,1)$ and in $L^2(0,1)$ ⊛

$\Rightarrow$ (uniqueness of limit) $\bar{u} = \hat{u}$ in L^2

$\Rightarrow \hat{u} = \bar{u} \in H^1(0,1)$

(iii) To show $\Phi(\tilde{f}) = \hat{u}$

$$\int_0^1 (\tilde{u})' v' + \tilde{f}(x) \tilde{u} v \, dx \overset{!}{=} \lim_{n\to\infty} \int_0^1 (\Phi(f_n))' v + \underset{\text{weak conv.} \uparrow \hat{f}}{f_n(x)} \underset{\text{cvg in } L^2 \uparrow \hat{u}}{\Phi(f_n)} v \, dx$$

$$= \int_0^1 s v \, dx \quad \forall v \in H_0^1(0,1)$$

⌈ Thm: H Hilbert space, $b \in L(H \times H, \mathbb{R})$

$$\left.\begin{array}{l} x_n \rightharpoonup x \\ y_n \to y \end{array}\right\} \Rightarrow b(x_n, y_n) \to b(x,y)$$
⌊

$\Rightarrow \hat{u} = \Phi(\hat{f})$

Def 2.1.H: For $g \in R(\Phi)$, $f \in \underset{f \in \mathcal{D}(\Phi)}{\arg\min} \{\|\Phi(f) - g\|_Y\} \neq \emptyset$

is called an <u>output least squares solution</u> $\Phi(f) = g$
With target $f^* \in X$ an output least squares solution that is closest to f^* is called an <u>f^*-output least squares solution</u>

2.2 Non-linear Tikhonov regularization

Recall Tikhonov reg. for $Tf = g^\delta$, $T \in \mathcal{L}(X,Y)$

$$R_\alpha g^\delta := \underset{f \in X}{\arg\min} \{\|Tf - g^\delta\|_Y + \alpha \|f\|_X^2\}, \quad \alpha > 0$$

→ Non-linear generalization for $\Phi(f) = g^\delta$

$$R_\alpha(g^\delta) := \underset{f \in \mathcal{D}(\Phi)}{\arg\min} \{\|\Phi(f) - g^\delta\|_Y^2 + \alpha \|f - f^*\|_X^2\}, \quad \alpha > 0$$

$$=: J_\alpha(f, g^\delta) \quad \text{target } f^* \in X$$

Lemma 2.2 C: X Hilbert space, $(x_n) \subset X$, $x_n \rightharpoonup x$

$$\|x\|_X \leq \liminf_{n\to\infty} \|x_n\|_X$$

Thm 2.2.B $\Phi : \mathcal{D}(\Phi) \subset X \to Y$ is weakly closed. Then for every $\alpha > 0$ a minimizer of (2.2.A) exists

Proof: $f \to J_\alpha(f, g^\delta)$ is bdd from below

$\Rightarrow \exists$ minimizing sequence $(f_n) \subset \mathcal{D}(\Phi)$

$$\|f_n\|_X \leq \|f_n - f^*\|_X + \|f^*\|_X \leq \sqrt{\frac{J_\alpha(f_n, g^\delta)}{\alpha}} + \|f^*\|_X < \infty$$

$$\|\Phi(f_n)\|_Y \leq \|\Phi(f_n) - g^\delta\|_Y + \|g^\delta\|_Y \leq \left(J_\alpha(f_n, g^\delta)\right)^{1/2} + \|g^\delta\|_Y < \infty$$

$\Rightarrow \exists$ weakly crg. subsequences (f_n), $(\Phi(f_n))$

$$\left.\begin{array}{l} f_n \rightharpoonup \tilde{f} \text{ in } X \\ \Phi(f_n) \rightharpoonup \tilde{g} \text{ in } Y \end{array}\right\} \overset{\text{weak closedness}}{\Longrightarrow} \quad \Phi(\tilde{f}) = \tilde{g}, \ \tilde{f} \in \mathcal{D}(\Phi)$$

To show: $\tilde{f}$ is a minimizer of the Tikhonov functional $J_\alpha(f, g^\delta)$

$$J_\alpha(\tilde{f}, g^\delta) \overset{(2.2.C)}{\leq} \liminf_{n\to\infty} \left\{ \|\Phi(f_n) - g^\delta\|_Y^2 + \alpha \|f_n - f^*\|_X \right\}$$

$$= \liminf_{n\to\infty} J_\alpha(f_n, g^\delta)$$

$$\big((f_n) \text{ minimizing sequence}\big) := \inf_{f \in \mathcal{D}(\Phi)} J_\alpha(f, g^\delta) \qquad \square$$

Goals:
- Show that "$R_\alpha : Y \to \mathcal{D}(\Phi)$" is continuous
- Convergence $R_\alpha(g^\delta) \to f^+$ as $\delta \to 0$, $\alpha \to 0$?
- Rate $\|R_\alpha(g^\delta) - f^+\|_X = O(\varphi(\delta))$, φ?

  a priori knowledge?

Thm 2.2.D, Continuity of R_α

$\Phi : \mathcal{D}(\Phi) \subset X \to Y$ weakly closed, $\alpha > 0$ fixed, $f^* \in X$ sequence $(g_n) \subset Y$, $g_n \to g^\delta$. Write $f_n \in \mathcal{D}(\Phi)$ for a minimizer of $f \to J_\alpha(f, g_n)$. Then $(f_n)_n \subset \mathcal{D}(\Phi)$ has a convergent subsequence and every convergent subsequence $\underbrace{\text{converges}}_{\text{in } X}$ to a minimizer of $f \to J_\alpha(f, g^\delta)$

<u>Proof:</u> for all $f \in \mathcal{D}(\Phi)$ (1)

$$\|\Phi(f_n) - g_n\|_Y^2 + \alpha \|f_n - f^*\|_X^2 = J_\alpha(f_n, g_n) \le J_\alpha(f, g_n)$$

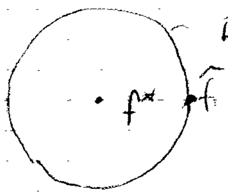
$\rightarrow (f_n) \subset X$, $(\Phi(f_n)) \subset Y$ bounded

$$\Rightarrow \begin{cases} f_n \rightharpoonup \hat{f} \in \mathcal{D}(\Phi) \\ \Phi(f_n) \rightharpoonup \hat{g} = \Phi(\hat{f}) \end{cases} \quad \text{[sub-sequencies !]}$$

As before: $\underset{\substack{\uparrow \\ \text{minimizer}}}{J_\alpha(\hat{f}, g^\delta)} \le \liminf\limits_{n \to \infty} J_\alpha(f_n, g_n)$

$$\overset{(1)}{\le} \lim_{n\to\infty} J_\alpha(f, g_n) = J_\alpha(f, g^\delta)$$

To show $f_n \to \hat{f}$ in X



$B_{\|f^* - f^\delta\|}(f^*)$ convex

In Hilbert space X, $C \subset X$ convex, closed, $(x_n) \subset C$, $x_n \rightharpoonup x \in C$

$\|x\|_X = \inf\limits_{y \in C} \|y\|_X \Rightarrow x_n \to x$

[Proof based on $\underset{\substack{x \neq y \\ \|x\| = \|y\| = 1}}{x, y \in H} \Rightarrow \|1/2\,(x+y)\| < 1$

Assume $f_n \not\to \hat{f} \Rightarrow \exists$ subsequence with $\|\underset{\nwarrow \text{subsequence}}{f_n} - f^*\|_X > \|\hat{f} - f^*\|_X$

$\lim\limits_{n\to\infty} \|\Phi(f_n) - g_n\|_Y^2 \leqq$