Series 1

1. Differentiation in $L^2_{\text{per}}([0,1])$

In class we investigated the inverse problem related to differentiation of continous periodic functions. In this exercise we investigate this challenge on a different pair of Hilbert spaces. Let

$$X := \left\{ f \in L^2_{\text{per}}([0,1])\,;\, \int_0^1 f\,dx = 0 \right\} \quad \text{and} \quad Y := \left\{ g \in H^1_{\text{per}}([0,1])\,;\, g(0) = 0 \right\}$$

be equipped with the L^2– and the H^1–norm respectively, and consider the direct operator $T : X \to Y$ defined by $(Tf)' = f$.

a) Show that T belongs to $\mathcal{L}(X,Y)$. Moreover, show that T^{-1} is in $\mathcal{L}(Y,X)$.

b) Show that the inverse problem associated to T is ill-posed, if the data g is perturbed with some noise, which can be controlled in the L^2–norm only.
`Hint:` Let $\mathcal{F} : X \to \ell^2(\mathbb{C})$ be the operator which, when evaluated on a function $v \in X$, returns the coefficient sequence $(\hat{v}_\ell)_{\ell\in\mathbb{Z}}$ of the Fourier series of v, that is,

$$\mathcal{F} = (\hat{v}_\ell)_{\ell\in\mathbb{Z}} = \left(\int_0^1 v(x) e^{-2\pi i \ell x}\,dx \right)_{\ell\in\mathbb{Z}} .$$

With the Parseval's identity it can be shown that $\mathcal{F}$ is an isometry. Characterize the operator $M : \ell^2(\mathbb{C}) \to \mathcal{F}(Y)$, which is the counterpart of T for Fourier series sequences, that is,

$$M := \mathcal{F} \circ T \circ \mathcal{F}^{-1} .$$

Then, consider a noisy data of the form

$$g^\delta := g + \delta e^{2\pi i k x} , \quad \text{for some } k \in \mathbb{N} ,$$

and, with the help of $\mathcal{F}$ and M, show that,

$$\|f - f^\delta\|_X = \Omega(k) , \text{ despite } \|g - g^\delta\|_X = \delta .$$

c) Show that the difference quotient operator

$$(R_h g)(x) := \frac{g(x+h) - g(x-h)}{2h} , \quad h > 0 ,$$

belongs to $\mathcal{L}(Z,X)$ for $Z = L^2_{\text{per}}(]0,1[)$ and that $\|R_h\|_{Z\to X} \to \infty$ for $h \to 0$.

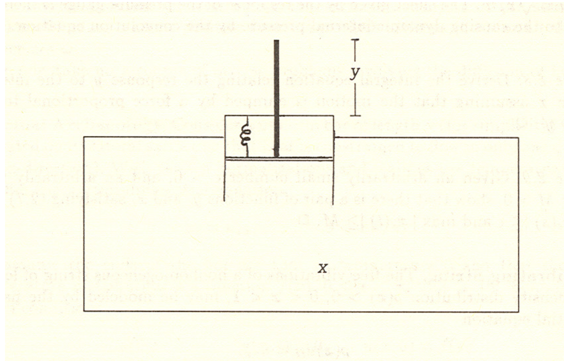

Abbildung 1: Sealed vessel with an attached gauge.

2. Pressure Gauges

Consider a sealed vessel with an attached gauge, see Fig. 2. The pressure in the vessel changes due to some external heating/cooling. The gauge is an undamped spring-loaded piston. The piston mass is m and the spring constant k. The cross-sectional area of the piston is 1 so that force and and pressure are equal. The gauge displacement $y(t)$ above the equilibrium and the internal dynamic pressure $x(t)$ then satisfy

$$my''(t) + ky(t) = x(t), \quad y(0) = 0, y'(0) = 0$$

with the initial conditions chosen for simplicity.

a) Using the Laplace transform, show that

$$y(t) = \frac{1}{\omega m} \int_0^t \sin(\omega(t-\tau))x(\tau)\, d\tau \quad \text{with} \quad \omega = \sqrt{k/m}. \tag{1}$$

b) Consider the inverse problem of determining the pressure $x(t)$ from the gauge displacement. Given an arbitrarily small $\varepsilon > 0$ and arbitrarily large $M > 0$, show that a pair of functions $y_\varepsilon(t), x_\varepsilon(t)$ exists satisfying (1) with $\max|y_\varepsilon(t)| \leq \varepsilon$ and $\max|x_\varepsilon(t)| \geq M$ for $t \in [0, T]$. One example suffices.

3. Deconvolution

In this exercise, we wish to implement a deconvolution similar to the gravimetry example from the lecture. We begin by theoretically deriving a method and then implement it and conduct some numerical experiments. Templates and some basic functions (e.g. for the generation of quadrature points) are available on the course website.

Consider the Hilbert spaces $X = Y = L^2([-1,1])$, the elements $f \in X$ and $g \in Y$ and the continuous linear operator $T : X \to Y$ defined by

$$T : f \mapsto \int_{-1}^{1} \mathcal{K}(\xi - x) f(x)\, \mathrm{d}x. \tag{2}$$

Siehe nächstes Blatt!

a) Show that T is linear and continuous for $\mathcal{K} \in L^1([-1,1])$.

b) Given $f(x) = \exp(x^2)$ and $\mathcal{K}(\sigma) = \exp(-\sigma^2)$, compute the analytical solution $g(\xi)$ to the *forward problem* $g = Tf$:

$$g(\xi) = \int_{-1}^{1} \mathcal{K}(\xi - x) f(x)\, \mathrm{d}x. \tag{3}$$

c) We now consider the *inverse problem* of determining $f \in X$ given a datum $g \in Y$. To this end, we introduce a reconstruction R_N with discretization parameter N based on a spectral Galerkin approach. This consists of approximating the unknown solution f by an element f_N of a finite-dimensional trial space V_N spanned by the $N+1$ basis functions $\{\varphi_j\}_{j=0}^{N}$

$$f_N(\xi) = \sum_{j=0}^{N} \mu_j \varphi_j(\xi). \tag{4}$$

Derive the weak formulation of (3) using V_N as the test space and the expansion of f from (4). Reformulate the problem as a linear system of equations $A\vec{\mu} = \vec{\psi}$.

d) As our choice for a concrete basis $\{\varphi_j\}$ we will use the $N+1$ Legendre polynomials $\{L_0(x), \ldots, L_N(x)\}$ on $[-1,1]$, defined by the recursion

$$(n+1)L_{n+1}(x) = (2n+1)xL_n(x) - nL_{n-1}(x)$$

with first two polynomials

$$L_0(x) = 1 \qquad L_1(x) = x.$$

Read up on the Legendre polynomials and their properties, eg. on Wikipedia.

e) Formulate (analytically) the quadrature method to approximate the entries A_{kj} with Gauss-Legendre quadrature using M points. For the trivial kernel $\mathcal{K} = 1$, what do you expect the matrix entries to be?

f) Write a Matlab function that accepts a function handle to $g(x)$ and the discretization parameter N and returns the coefficient vector $\vec{\mu}$ that solves the linear system derived above, i.e.:

```
function mu = IP_spectral_galerkin(g,K,N)
  % solve inverse problem using spectral Galerkin approach
  %   g: function handle representing the data
  %   K: function handle representing the kernel
  %   N: discretization param. Number of basis fcts is N+1
  %   mu: coefficient vector of Lagrange basis of length N+1
end
```

You may use the Matlab function `gauleg(a,b,M)` from the file `gauleg.m` to construct the Gauss-Legendre quadrature rule with M points for an integral over the interval $[a,b]$. $M = 20$ is a good value. The evaluation of the Legendre basis at all points in the row vector `x` is implemented in the function `legendre(N,x)` in the file `legendre.m`. See the course website for a download link for these files.

Bitte wenden!

g) Write a Matlab function that accepts the computed coefficient vector of length $N+1$ and a vector of (many) evaluation points and computes the solution f_N at these points, i.e.:

```
function f = eval_legendre(mu,x)
  % mu: coefficient vector of Legendre basis
  % x: vector of evaluation points of length
  % f: value of f_N at all points in the vector x
end
```

`Hint:` The solution should look like the following:

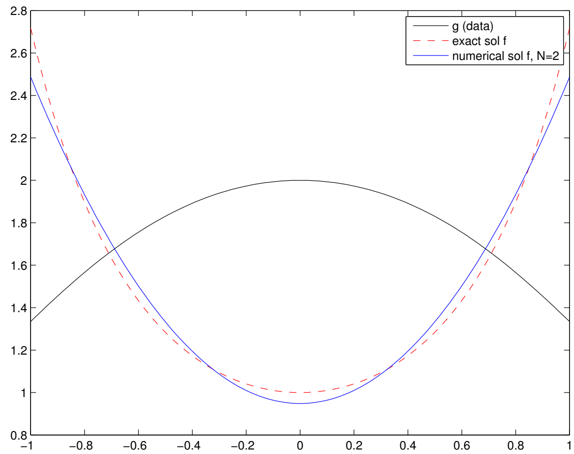

Abbildung 2: Solution of the inverse problem with $N = 2$ (3 basis functions L_0, L_1, L_2).

h) Use overkill quadrature (eg. Gaussian quadrature with 10^4 points) to compute the L^2-norm of the discretization error $e_N(x) = f(x) - R_N(g)$ caused by the use of a finite-dimensional trial space V_N. Plot this error against N for $N = 1, 2, \dots, 15$. What kind of convergence do you observe? Is this expected? Measure the convergence rate.

i) Consider the perturbed data $g^\delta(x) = g(x) + \delta \sin(n\pi x)$, where δ is the noise parameter. Decompose the total error $f - R_N(g^\delta)$ into a reconstruction error and a data noise error. What convergence behavior in N of the total error with fixed δ do you expect?

j) For each fixed $\delta = 10^{-k}, k = 0, 2, 4, 6, 8, 10$, compute the numerical approximation of the solution to the inverse problem and use overkill quadrature (eg. Gaussian quadrature with 10^4 points) to compute the L^2-norm of the total error $e_N^\delta = f(x) - R_N(g^\delta)$ and plot it vs. N for $N = 1, 2, \dots, 12$ for $n = 1$. Can you verify your prediction from subproblem i)?

k) In order to gain more insight into the divergence of the reconstruction operator norm, plot the 2-norm condition of the matrix A vs. N for $N = 1, 2, \dots, 12$. How does the condition grow as a function of N?