Series 2

1. Symm's Integral Equation

The single layer potential[1]

$$u(\mathbf{x}) = -\frac{1}{2\pi}\int_{\partial\Omega} f(\mathbf{y})\log|\mathbf{x}-\mathbf{y}|\,ds(\mathbf{y})\,,\quad \mathbf{x}\in\Omega$$

solves the boundary value problem

$$\Delta u = 0 \quad \text{in } \Omega\,,\quad u = g \quad \text{on } \partial\Omega,$$

where $\Omega\subset\mathbb{R}^2$, if and only if the density $f\in C(\partial\Omega)$ solves the *Symm's equation*

$$-\frac{1}{2\pi}\int_{\partial\Omega} f(\mathbf{y})\log|\mathbf{x}-\mathbf{y}|\,ds(\mathbf{y}) = g(\mathbf{x})\,,\quad \mathbf{x}\in\partial\Omega\,. \tag{1}$$

We consider the special case that $\partial\Omega$ is a circle of radius 1.

a) Show that (1) is equivalent to

$$(T\tilde{f})(\zeta) = \tilde{g}(\zeta)\,,\quad \zeta\in[0,2\pi[\,,$$

where $\tilde{f}(\zeta) := f(\cos(\zeta),\sin(\zeta))$, $\tilde{g}(\zeta) := g(\cos(\zeta),\sin(\zeta))$, and

$$(T\tilde{f})(\zeta) := -\frac{1}{2\pi}\int_0^{2\pi}\tilde{f}(\eta)\log\left(2\left|\sin\left(\frac{\zeta-\eta}{2}\right)\right|\right)\,d\eta\,. \tag{2}$$

`Hint:` use a parametrization of the circle.

b) Let $X = L^2_{\text{per}}([0,2\pi[)$ be the space of square integrable periodic complex functions. Show that (2) defines a linear self-adjoint compact operator $T : X\to X$ and characterize its spectrum.
`Hint:` Show that the kernel of T is in $L^2([0,2\pi[^2)$.

c) Show that T commutes with the shift operator $(Sf)(z) := f(z-s)$, for $s>0$ fixed.

d) Show that $S : X\to X$ is a unitary operator which admits the expansion

$$(Sf)(\zeta) = \sum_{n\in\mathbb{Z}} e^{-ins}(f, e^{in\zeta})_X e^{in\zeta}\,.$$

`Hint:` Show that $e^{in\zeta}$ is an eigenfunction of S.

[1] Throughout the exercise, log denotes the natural logarithm.

e) Show that $e^{in\zeta}$ is an eigenfunction of T , and compute the corresponding eigenvalue. Conclude that all eigenfuctions of T are of the form $Ae^{in\zeta} + Be^{-in\zeta}$. What is the SVD of T?

2. Numerical collocation method for Abel's integral equation

Consider the Abel integral operator of the first kind $T : f \mapsto g$ defined as

$$g(x) = (Tf)(x) := \frac{1}{\sqrt{\pi}} \int_0^x \frac{f(y)}{\sqrt{x-y}} \,\mathrm{d}y, \qquad x \geq 0. \tag{3}$$

For $g \in H^1([0,1]), g(0) = 0$, (3) has a unique solution $f^+ = T^+ g \in L^2([0,1])$, namely

$$f^+(x) = \frac{1}{\sqrt{\pi}} \int_0^x \frac{g'(t)}{\sqrt{x-t}} \,\mathrm{d}t \quad \forall x \in [0,1]. \tag{4}$$

We wish to numerically approximate this solution (instead of using formula (4)[2]) by a function f_N using a collocation method. Given an equidistant discretization of $[0,1]$ with $N+1$ points, $\mathcal{M}_N := \{0, h, 2h, \dots, Nh\}$, we consider the trial space $\mathcal{S}_1^0(\mathcal{M}_N)$ consisting of globally continuous, piecewise linear functions.

For $g \in H^1([0,1])$ and $g(0) = 0$ one can show that $f(0) = 0$. We can now formulate the collocation conditions for this problem, namely

$$Tf_N(x_i) = g(x_i) \quad \forall x_i \in \mathcal{M}_N. \tag{5}$$

Denoting by f_i an approximation of $f(x_i), i \geq 0$ and writing $f_N \in \mathcal{S}_1^0(\mathcal{M}_N)$ as

$$f_N(y) = \frac{1}{h} \left[(y - ih) f_{i+1} + ((i+1)h - y) f_i \right] \quad \text{for } ih < y \leq (i+1)h, \tag{6}$$

we can formulate a discretized version of (3):

$$g(ih) = \frac{1}{\sqrt{\pi}} \int_0^{ih} \frac{f_N(y)}{\sqrt{ih-y}} \,\mathrm{d}y, \qquad i \geq 1.$$

a) By splitting the integral into a sum of integrals over the grid segments $(jh, (j+1)h)$ and rescaling the variable of integration so that the integration interval of each summand is $(0,1)$, show that

$$g(ih) = \frac{\sqrt{h}}{\sqrt{\pi}} \sum_{j=0}^{i-1} \int_0^1 \frac{f_N((j+s)h)}{\sqrt{i-j-s}} \,\mathrm{d}s. \tag{7}$$

[2]Note that an inversion formula is often not available for the generalized problem $g(x) = \int_0^x \frac{k(x,y)}{\sqrt{x-y}} f(y) dy$, where $k(x,y)$ is some smooth function.

Gauss Quadrature We now use a one-point weighted Gaussian quadrature rule with weight w_k and node ξ_k,

$$\int_0^1 \frac{\varphi(s)}{\sqrt{k-s}}\,\mathrm{d}s \approx w_k \varphi(\xi_k). \tag{8}$$

b) Compute analytically the weight and node for this rule. Implement a Matlab function to compute these two quantities (needed later).
`Hint:` An n-point Gaussian quadrature rule should exactly integrate polynomials of order $2n-1$.

c) Implement a function that computes all f_i, $i > 0$ given f_0.
`Hint:` After inserting the quadrature rule from above, computing f_i only requires the values $f_j, j < i$; thus, simple forward substitution solves the problem.

d) Apply the code from above for $g(x) = x^3/3$ and $N = 50$. Compute the exact solution with (4) and plot the error vs. $N = 2^{(1:8)}$. Check the second order L^2 convergence of discretization error:

$$|f_i - f(ih)| \leq CN^{-2},$$

which holds if $f \in L^2([0,1])$.

e) To investigate the effects of a measurement error, we assume our data to be perturbed like

$$(g^\delta - g)(x) \in \{\delta \sin(k\pi x) \,:\, 1 \leq k \leq N-1\}. \tag{9}$$

For $\delta \in \{10^{-1}, 10^{-2}, 10^{-3}\}$, compute the solution using the method above for the data g^δ. Compute the error as the maximum over all perturbations in the set (9) (this is a sort of discrete analog of the supremum in the definition of a *deterministic regularization method*, see Def. 1.3.3.A from the lecture). For each δ, plot the L^2 error vs. N for the same values as above. What do you observe?

Tikhonov Regularization In addition to the regularization by discretization applied above, we wish to use Tikhonov regularization. Consider the Tikhonov functional applied to the discrete problem

$$J_\alpha = ||A_N f_N - I_N g||^2_{\mathbb{R}^N} + \alpha ||f_N||^2_{L^2}, \quad \alpha > 0, \tag{10}$$

where A_N is the finite-dimensional operator, $f_N \in \mathcal{S}_1^0(\mathcal{M}_N)$ is the unknown solution and I_N is an interpolation operator with respect to the grid $\mathcal{M}_N$. In terms of the coefficient vector $\vec{f}_N = (f_1, \ldots, f_N)^T \in \mathbb{R}^N$, we can write the problem as

$$||\mathbf{A}_N \vec{f}_N - I_N g||^2_{\mathbb{R}^N} + \alpha \vec{f}_N^T \mathbf{M}_N \vec{f}_N \to \min, \quad \alpha > 0,$$

with the system matrix $\mathbf{A}_N$ and mass matrix $\mathbf{M}_N$. This leads to the extended normal equations

$$(\mathbf{A}_N^T \mathbf{A}_N + \alpha \mathbf{M}_N)\vec{f}_N = \mathbf{A}_N^T I_N g. \tag{11}$$

f) In order to solve (11) we require the system matrix $\mathbf{A}_N$. Based on the sum derived above, derive the elements of the system matrix $\mathbf{A}_N$ and implement a function to assemble it. Also implement the mass matrix $\mathbf{M}_N$, which is just the standard finite elements mass matrix.
`Hint 1:` Since $g_0 = f_0 = 0$, you can just ignore the index 0.
`Hint 2:` Show that $\mathbf{A}_N$ is lower triangular and Toeplitz. See `toeplitz(a,b)`.
`Hint 3:` Look closely at the last basis function when constructing $\mathbf{M}_N$.

g) Write a function that solves (11) and plot the L^2-error vs. N and α for $N = 2^1 \dots 2^{10}$, $\alpha = 2^{-1}, \dots, 2^{-16}$ (use `surf` to visualize it as a surface over the N-α-plane). How does the error compare to the one computed in (d)?

h) We again wish to investigate the effects of a measurement error and use the same model as above, (9). For $\delta \in \{10^{-1}, 10^{-2}, 10^{-3}\}$, compute the solution using Tikhonov regularization as above for the data g^δ. Again compute the error as the maximum over all perturbations in the set (9) For each δ, plot the L^2 error vs. N and α for the same values as above. What do you observe? Compare the errors to the case where only regularization by discretization was applied – does Tikhonov regularization improve anything?

i) For a fixed noise level δ, determine empirically the best value for N and α (e.g. by iterating over all computed error values). Plot the error for these optimal values of the parameters vs. δ. What convergence rate (in δ) do you observe?