Series 3

1. Numerical Differentiation and Discrepancy Principle

In class we saw that the discrepancy principle can equip regularization methods with convergence properties with respect to the noise level δ. The goal of this exercise is to see the discrepancy principle in action. As a test case, we consider the problem of differentiating continuous periodic functions. Let

$$X := \left\{ f \in L^2_{\text{per}}([0,1])\,;\ \int_0^1 f\,dx = 0 \right\}$$

be equipped with the L^2–norm, and consider the direct operator $T : X \to L^2_{\text{per}}([0,1])$ defined by $(Tf)' = f$. As we already know, this leads to an ill-posed problem. The first task is to provide a suitable discretization of the test case.

a) A natural choice is to consider a Ritz-Galerkin discretization in L^2 on the finite element space spanned by the derivatives of hat functions. Implement a function that computes the finite element solution f_h of

$$Tf(x) = \sin(2\pi x)\,,$$

and investigate the convergence of f_h to the exact solution in the L^2–norm.

b) To show that the discretized problem is nevertheless ill-posed, plot the L^2–norm of the finite element solution of

$$Tf(x) = \sin(2k\pi x)$$

for `k=10:10:200`.

c) Compute the Tikhonov solution of

$$Tf(x) = \sin(2\pi x) + \delta \frac{\chi_{[0.4,0.6]}}{\sqrt{2}}$$

with $\delta = 0.1$ and plot the total error as a function of $\alpha = 2^{-1,\ldots,-12}$.

d) Compute the Landweber solution of

$$Tf(x) = \sin(2\pi x) + \delta \frac{\chi_{[0.4,0.6]}}{\sqrt{2}}$$

with $\delta = 0.1$ and plot the total error as a function of the number of iterations `R=25:25:500`.
`Hint:` to compute the iteration step μ, estimate the norm $\|T^*T\|$ with a power method.