Series 4

In the last exercise, you implemented various regularization methods for the direct operator T defined by $(Tf)' = f$ (see Series 3 for details). We will now consider one more method, CG, and then use the discrepancy principle as an a-posteriori parameter choice rule.

1. Conjugate Gradient Method

Implement the Conjugate Gradient method for solving $T^*Tf = T^*g^\delta$.

Algorithm 1 Conjugate Gradient Algorithm for $T^*Tf = T^*g^\delta$

$f_0 = 0, r_0 = T^*g^\delta, p_0 = r_0, k = 0$
while $||g^\delta - Tf_k||_Y > tol$ **do**
  $\alpha = (r_k, r_k)_X / (Tp_k, Tr_k)_Y$
  $f_{k+1} = f_k + \alpha p_k$
  $r_{k+1} = r_k - \alpha T^*Tp_k$
  $\beta = (Tr_{k+1}, Tp_k)_Y / (Tp_k, Tp_k)_Y$
  $p_{k+1} = r_{k+1} - \beta p_k$
  $k = k + 1$
end while

2. Discrepancy Principle

For all exercises below, consider the perturbed data $g^\delta(x) = \sin(2\pi x) + \delta \frac{\chi_{[0.4,0.6]}(x)}{\sqrt{2}}$.
A common choice for the parameter τ in the discrepancy principle is 2.

a) Tikhonov Implement the Newton method elucidated in Section 1.4.2 of the lecture to compute the regularization parameter α_d. For $\delta \in \{0.1, 0.05, 0.01, 0.005, 0.001\}$, compute the Tikhonov solution using α_d and plot the error vs δ.

b) Landweber Modify your implementation of the Landweber iteration to stop once the condition imposed by the discrepancy principle is met, i.e. $||g^\delta - T\hat{f}_k|| \le \tau\delta$. Plot the error vs. δ.

c) Conjugate Gradients Modify your implementation of the Conjugate Gradient method to stop once the condition imposed by the discrepancy principle is met, i.e. $||g^\delta - T\hat{f}_k|| \le \tau\delta$. Plot the error vs. δ.