Exercise Sheet 3

Exercise 1

(i) Let G be a topological group and let H be a subgroup of G. Equip G/H with the quotient topology. Show that G is connected if both H and G/H are connected.

(ii) Show that $\mathrm{SO}(n)$ is connected for all $n \in \mathbb{N}$.

A similar argument works for $\mathrm{SU}(n)$.

Exercise 2

Let $G = \mathrm{SL}(2,\mathbb{R})$ act on $\mathbb{H} = \{x + iy \in \mathbb{C} \mid y > 0\}$ via fractional linear transformations. Show that this action is transitive, $\mathrm{Stab}_G(i) = \mathrm{SO}(2,\mathbb{R}) =: H$ and that the induced isomorphism of G-spaces $\mathbb{H} \cong G/H$ is a homeomorphism.

In general, let G be a Lie group acting smoothly and transitively on a manifold X. Then for every $x \in X$ there is a unique differential structure on $G/\mathrm{Stab}_G(x)$ such that the natural map $G/\mathrm{Stab}_G(x) \cong X$ is a G-diffeomorphism.

Exercise 3

Consider the action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{P}^1\mathbb{R} = \{V \leq \mathbb{R}^2 \text{ subspace} \mid \dim V = 1\}$ given by $g_*V := gV$ $(g \in \mathrm{SL}(2,\mathbb{R}),\ V \in \mathbb{P}^1\mathbb{R})$. Show directly that there is no Radon measure, which is non-zero on non-empty open sets, on $\mathbb{P}^1\mathbb{R}$ that is invariant under the above action of $\mathrm{SL}(2,\mathbb{R})$.

Convince yourself that this also follows from the according theorem of the lecture.

Exercise 4

Let G be a *compact* Hausdorff group with left Haar measure μ_G and let $H \leq G$ be a closed subgroup of G with left Haar measure μ_H. Show that the proof of the theorem on the existence of quotient measures produces the following G-invariant Radon measure $\mu_{G/H}$ on G/H: For all $E \subseteq G/H$ measurable,

$$\mu_{G/H}(E) = \frac{\mu_G(p^{-1}(E))}{\mu_H(H)}, \text{ in particular } \mu_{G/H}(G/H) = \frac{\mu_G(G)}{\mu_H(H)}.$$

This is an "honest" quotient measure. Think about what goes wrong in the case when G is not compact and how the proof mends it.