Exercise Sheet 4

Exercise 1

Let $p, q \in \mathbb{N}_0$. Show that $\mathrm{O}(p,q)$ is a Lie group.

Exercise 2

Let M be a smooth manifold and let $p \in M$. Consider the set

$$F(p) = \{(U, f) \mid U \in \mathcal{U}(p) \text{ open}, \ f : U \to \mathbb{R} \text{ smooth}\}.$$

(i) Show that the relation $\sim$ on $F(p)$ defined by

$$(U_1, f_1) \sim (U_2, f_2) \ \Leftrightarrow \ \exists\, U_3 \in \mathcal{U}(p), \text{ open}, \ U_3 \subseteq U_1 \cap U_2 : \ f_1|_{U_3} \equiv f_2|_{U_3}$$

is an equivalence relation.

(ii) Show that $C^\infty(p) := F(p)/\sim$ is an algebra over $\mathbb{R}$ with the operations induced from $F(p)$.

If $\mathcal{F}$ is the sheaf of smooth functions on M, then $C^\infty(p)$ is the stalk of $\mathcal{F}$ at p.

Exercise 3

Let M be a manifold and let $p \in M$. Consider the tangent space of M at p:

$$T_p(M) := \{X_p : C^\infty(p) \to \mathbb{R} \mid X_p \text{ is linear}, \ X_p(fg) = f(p)X_p(g) + g(p)X_p(f)\}.$$

Further, let (U, φ) be a chart of M at p such that $\varphi(p) = 0$. Show that the map

$$\alpha : \mathbb{R}^n \to T_pM, \ v \mapsto \big(f \mapsto D_0(f \circ \varphi^{-1})v\big)$$

is an isomorphism of vector spaces.

Note that if $v = D_0(\varphi \circ \gamma)$, where $\gamma : (-1, 1) \to M$ such that $\gamma(0) = p$, is a tangent vector in the usual sense, then $\alpha(v)(f) = D_0(f \circ \gamma)$ is the directional derivative of f in the direction v.