Exercise Sheet 5

Exercise 1

Show that the Lie algebra $(\mathbb{R}^3, \times)$, where $\times$ denotes the cross product, is isomorphic to the Lie algebra of $\mathrm{O}(3, \mathbb{R})$.

This Lie algebra is 3-dimensional. There is just one isomorphism class of 1*-dimensional Lie algebras; and there are exactly two isomorphism classes of* 2*-dimensional Lie algebras. However, there are uncountably many pairwise non-isomorphic 3-dimensional Lie algebras.*

Exercise 2

Let G be a Lie group and let M be a smooth manifold. Further, let G act smoothly and transitively on M. Fix $m_0 \in M$.

(i) Identify $\mathrm{Vect}^\infty(M)^G$ with a subspace of $T_{m_0}M$.

(ii) Show that there are no non-zero $\mathrm{O}(3, \mathbb{R})$-invariant vector fields on $S^2 \subseteq \mathbb{R}^3$.

Interpret (i) in the case of the left multiplication action of a Lie group on itself.

Exercise 3

Let M be a smooth manifold. Further, let X and Y be smooth complete vector fields on M and denote by $\Phi^X : \mathbb{R} \times M \to M$ and $\Phi^Y : \mathbb{R} \times M \to M$ the corresponding flows. Then the following statements are equivalent:

(i) $[X, Y] = 0$.

(ii) For all $s, t \in \mathbb{R},\ m \in M$: $\Phi_t^X \circ \Phi_s^Y(m) = \Phi_s^Y \circ \Phi_t^X(m)$.

Here, we use the notation $\Phi_t^X(m) := \Phi^X(t, m)$ and $\Phi_s^Y(m) := \Phi^Y(s, m)$.

Condition (ii) depicts nicely what it means for two vector fields to have a zero bracket.