Exercise Sheet 7

Exercise 1

Let Γ be a non-zero discrete subgroup of $\mathbb{R}^n$, $n \in \mathbb{N}$. Show there are an integer $r \in \{1, \ldots, n\}$ and vectors $(v_1, \ldots, v_r)$ in $\mathbb{R}^n$ such that $\Gamma = \mathbb{Z}\, v_1 \oplus \mathbb{Z}\, v_2 \oplus \cdots \oplus \mathbb{Z}\, v_r$.

What are the discrete subgroups of $\mathbb{R}^n$ such that the quotient has finite "Haar" measure? These subgroups are called lattices *in $\mathbb{R}^n$.*

Exercise 2

Consider the Heisenberg group

$$G := \left\{ \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \middle| \, x, y, z \in \mathbb{R} \right\} \subseteq \mathrm{GL}(3, \mathbb{R})$$

and the subgroup

$$H := \left\{ \begin{pmatrix} 1 & 0 & m \\ & 1 & 0 \\ & & 1 \end{pmatrix} \middle| \, m \in \mathbb{Z} \right\}$$

of G. Check that G/H is a connected, solvable Lie group and show that G/H does not admit a smooth, injective homomorphism into $\mathrm{GL}(V)$ for any finite-dimensional $\mathbb{C}$-vector space V.

This shows that Lie's Theorem does not immediately provide an understanding of all solvable connected Lie groups.

Exercise 3

Let G be a Lie group. Define $G_{(1)} := [G, G]$ and $G_{(k)} := [G, G_{(k-1)}]$ for all $k \in \mathbb{N}_{\geq 2}$. Then G is called *nilpotent* if $G_{(n)} = \{e\}$ for some $n \in \mathbb{N}$. Show that every nilpotent Lie group is solvable and give an example of a solvable Lie group which is not nilpotent.

You may want to look up the analogue of Lie's theorem for nilpotent Lie groups.