ETH Zürich HS 2013
D-MATH
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# Mathematical Finance

## Exercise Sheet 0

*This exercise sheet introduces stochastic calculus for general (possibly discontinuous) semimartingales, which will be used throughout the course.*

### Exercise 0-1

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. Denote by $\mathcal{H}^2$ the set of all RCLL martingales which are bounded in $L^2$, i.e, satisfy $\sup_{t\geq 0} \mathbb{E}[M_t^2] < \infty$. Recall that $\mathcal{H}^2$ becomes a Hilbert space when endowed with the norm $\|M\| := \|M_\infty\|_{L^2(\mathbb{P})}$. It can be shown that $\mathcal{H}_0^2 := \{M \in \mathcal{H}^2 : M_0 = 0\}$ and $\mathcal{H}_0^{2,c} := \{M \in \mathcal{H}_0^2 : M \text{ continuous}\}$ and $\mathcal{H}_0^{2,d} = (\mathcal{H}_0^{2,c})^\times := \{M \in \mathcal{H}_0^2 : \mathbb{E}[M_\infty N_\infty] = 0 \text{ for all } N \in \mathcal{H}_0^{2,c}\}$ are closed linear subspaces and *stable under stopping*. Each $M \in \mathcal{H}^2$ can be uniquely decomposed as $M = M_0 + M^c + M^d$, where $M^c \in \mathcal{H}_0^{2,c}$ and $M^d \in \mathcal{H}_0^{2,d}$. Denote the *localised versions* of the above spaces by $\mathcal{H}^2_{\text{loc}}$, $\mathcal{H}^2_{0,\text{loc}}$, $\mathcal{H}^{2,c}_{\text{loc}}$ and $\mathcal{H}^{2,d}_{0,\text{loc}}$. Each $M \in \mathcal{H}_0^{2,d}$ is called a *purely discontinuous* $L^2$-bounded martingale, and it can be shown that $M \in \mathcal{H}_0^2$ is in $\mathcal{H}_0^{2,d}$ if and only if $\mathbb{E}[M_\infty^2] = \mathbb{E}[\sum_{s>0}(\Delta M_s)^2]$. One can show that there exists for each $M \in \mathcal{H}_0^2$ a unique adapted, increasing, RCLL process $[M] = ([M]_t)_{t\geq 0}$ null at 0 with $\Delta[M] = (\Delta M)^2$ and such that $M^2 - [M]$ is a uniformly integrable martingale null at 0. $[M]$ is called the *(optional) quadratic variation of* $M$. For $L, M \in \mathcal{H}_0^2$, the *covariation of* $L$ *and* $M$ is defined via polarisation by $[L, M] := \frac{1}{4}([L + M] - [L - M])$. $[\cdot, \cdot]$ satisfies the natural consistency properties with respect to *stopping*, i.e., $[L, M]^\tau = [L, M^\tau]$ for each stopping time $\tau$ and $L, M \in \mathcal{H}_0^2$, and this is used to extend the definition to $L, M \in \mathcal{H}^2_{0,\text{loc}}$.

(a) Let $L \in \mathcal{H}_0^{2,c}$ and $M \in \mathcal{H}_0^{2,d}$. Show that $[L, M] \equiv 0$.
*Hint:* Show that $LM$ is a uniformly integrable martingale and that $[L, M]$ is continuous.

(b) Let $L, M \in \mathcal{H}^2_{0,\text{loc}}$ be arbitrary. Show that

$$[L, M] = \langle L^c, M^c\rangle + [L^d, M^d] = \langle L^c, M^c\rangle + \sum_{0<s\leq\cdot} \Delta L_s \Delta M_s.$$

(c) Let $N = (N_t)_{t\geq 0}$ be a *Poisson process* with rate $\lambda > 0$ and $(Y_k)_{k\geq 1}$ a sequence of random variables independent of $N$ and such that the $Y_k$ are i.i.d., square-integrable with mean $\mu$ and $\mathbb{P}[Y_k = 0] = 0$. Define the *compensated compound Poisson process* $X = (X_t)_{t\geq 0}$ by

$$X_t := \sum_{k=1}^{N_t} Y_k - \mu\lambda t,$$

and assume about the filtration that $X$ is a Lévy process with respect to $(\mathscr{F}_t)_{t\geq 0}$. (This is for instance satisfied if the filtration is generated by $X$.) Show that $X \in \mathcal{H}^{2,d}_{0,\text{loc}}$ and $[X]_t = \sum_{k=1}^{N_t} Y_k^2$.
*Hint:* For $n \in \mathbb{N}$, denote by $\sigma_n := \inf\{t \geq 0 : N_t = n\}$ the $n$-th jump time of the Poisson process. The elementary theory of Poisson processes shows that $\sigma_n$ is Gamma$(n, \lambda)$-distributed. In particular, $\mathbb{E}[\sigma_n] = \frac{n}{\lambda}$ and $\operatorname{Var}(\sigma_n) = \frac{n}{\lambda^2}$, $n \in \mathbb{N}$.

### Exercise 0-2

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. An adapted RCLL process $X = (X_t)_{t\geq 0}$ is called a *semimartingale* if it can be (**not** uniquely) decomposed as $X = X_0 + M + A$ with $M \in \mathcal{H}^2_{0,\text{loc}}$ and $A$ adapted, RCLL and of finite variation. (In the usual definition of a semimartingale, $M$ is only required to be a local martingale. However, one can show that both definitions are equivalent.)

Decomposing $M$ as $M = M^c + M^d$ with $M^c \in \mathcal{H}^{2,c}_{0,\mathrm{loc}}$ and $M^d \in \mathcal{H}^{2,d}_{0,\mathrm{loc}}$, we set $X^c := M^c$, and call $X^c$ the *continuous local martingale part of* $X$.

(a) Show that $X^c$ is well defined, in the sense that if $X_0 + M + A$ and $X_0 + \widetilde{M} + \widetilde{A}$ are two different decompositions of $X$ with $M, \widetilde{M} \in$, then $M^c = \widetilde{M}^c$ $\mathbb{P}$-a.s.

*Hint:* Use without proof that every $L \in \mathcal{H}^2_{0,\mathrm{loc}}$ of finite variation is in $\mathcal{H}^{2,d}_{0,\mathrm{loc}}$.

In order to define a *stochastic integral* with respect to a general semimartingale $X$, one defines – similarly to the continuous case – first a stochastic integral for locally square-integrable martingales. To this end, fix $M \in \mathcal{H}^2_{0,\mathrm{loc}}$, define

$$L^2(M) := \left\{ H \text{ predictable} : \mathbb{E}\left[ \int_0^\infty H_s^2 \,\mathrm{d}[M]_s \right] < \infty \right\},$$

and denote by $L^2_{\mathrm{loc}}(M)$ its *localised version*. One can show that for each $H \in L^2(M)$, there exists a unique element $H \bullet M$ of $\mathcal{H}^2_0$ satisfying

$$[H \bullet M, L] = \int_0^\cdot H_s \,\mathrm{d}[M, L]_s \quad \forall L \in \mathcal{H}^2_0.$$

$H \bullet M$ is called the *stochastic integral of* $H$ *with respect to* $M$. It satisfies the natural consistency properties with respect to *stopping*, i.e., $(H \bullet M)^\tau = H\mathbb{1}_{((0,\tau]]} \bullet M = H \bullet M^\tau$, and this is used to extend the definition to $H \in L^2_{\mathrm{loc}}(M)$.

(b) Let $M \in \mathcal{H}^2_{0,\mathrm{loc}}$ and $H \in L^2_{\mathrm{loc}}(M)$. Show that if $M \in \mathcal{H}^{2,c}_{0,\mathrm{loc}}$ or $M \in \mathcal{H}^{2,d}_{0,\mathrm{loc}}$, then also $H \bullet M \in \mathcal{H}^{2,c}_{0,\mathrm{loc}}$ or $H \bullet M \in \mathcal{H}^{2,d}_{0,\mathrm{loc}}$, respectively, and that in the first case $H \bullet M$ coincides with the stochastic integral $\int_0^\cdot H_s \,\mathrm{d}M_s$ from the course BMSC.

If $A$ is adapted and of finite variation, denote by $L(A)$ all predictable processes which are (path-by-path) Lebesgue–Stieltjes integrable with respect to $A$. If $X$ is a general semimartingale, set

$$L(X) := \Big\{ H \text{ predictable} : \text{there exists a decomposition } X = X_0 + M + A \text{ such that } H \in L^2_{\mathrm{loc}}(M) \cap L(A) \Big\},$$

and for $H \in L(X)$, define the stochastic integral of $H$ with respect to $X$ by

$$H \bullet X := H \bullet M + \int_0^\cdot H_s \,\mathrm{d}A_s, \quad \text{where } X = X_0 + M + A \text{ and } H \in L^2_{\mathrm{loc}}(M) \cap L(A).$$

One can show that $H \bullet X$ is well defined, in the sense that if $X_0 + M + A$ and $X_0 + \widetilde{M} + \widetilde{A}$ are two different decompositions of $X$ with $H \in L^2_{\mathrm{loc}}(M) \cap L(A)$ and $H \in L^2_{\mathrm{loc}}(\widetilde{M}) \cap L(\widetilde{A})$ then $H \bullet M + \int_0^\cdot H_s \,\mathrm{d}A_s = H \bullet \widetilde{M} + \int_0^\cdot H_s \,\mathrm{d}\widetilde{A}_s$. (Moreover, one can show that our definition of $L(X)$ coincides with the usual one, which is beyond the scope of our course.)

(c) Let $X$ and $Y$ be semimartingales. Show that $Y_- \in L(X)$.

**Exercise 0-3**

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. For a semimartingale $X$, the *quadratic variation of* $X$ is defined as $[X]_t := \langle X^c \rangle + \sum_{0<s\leq \cdot} (\Delta X_s)^2$, and it can be shown that the infinite series converges $\mathbb{P}$-a.s. For semimartingales $X$ and $Y$, the *quadratic covariation of* $X$ *and* $Y$ is defined via polarisation, and it is not difficult to check that $[X, Y] = \langle X^c, Y^c \rangle + \sum_{0<s\leq \cdot} \Delta X_s \Delta Y_s$. Moreover, the product $XY$ is again a semimartingale and satisfies the *product rule*

$$XY = X_0 Y_0 + X_- \bullet Y + Y_- \bullet X + [X, Y].$$

Finally, if $X$ is a semimartingale and $f : \mathbb{R} \to \mathbb{R}$ is in $C^2$, then $f(X)$ is again a semimartingale and satisfies *Itô's formula*

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_{s-})\,\mathrm{d}X_s + \int_0^t \frac{1}{2} f''(X_{s-})\,\mathrm{d}[X]_s \\ &\quad + \sum_{0<s\le t} \Big(\Delta f(X_s) - f'(X_{s-})\Delta X_s - \frac{1}{2} f''(X_{s-})(\Delta X_s)^2\Big) \\ &= f(X_0) + \int_0^t f'(X_{s-})\,\mathrm{d}X_s + \int_0^t \frac{1}{2} f''(X_{s-})\,\mathrm{d}\langle X^c\rangle_s \\ &\quad + \sum_{0<s\le t} \Big(\Delta f(X_s) - f'(X_{s-})\Delta X_s\Big). \end{aligned}$$

(a) Let $X$ be a semimartingale with $X_0 := 0$. Define the process $Z = (Z_t)_{t\ge0}$ by

$$Z_t := \exp\left(X_t - \frac{1}{2}\langle X^c\rangle_t\right) \prod_{0<s\le t} (1 + \Delta X_s)\exp(-\Delta X_s).$$

Show that $Z_t$ is well defined for all $t \ge 0$, that $Z$ is a semimartingale and that it satisfies the SDE $Z_t = 1 + \int_0^t Z_{s-}\,\mathrm{d}X_s$. $Z$ is called the *stochastic exponential of* $X$ and denoted by $\mathcal{E}(X)$.

*Hint:* Define the processes $Y = (Y_t)_{t\ge0}$ and $A = (A_t)_{t\ge0}$ by $Y_t := X_t - \frac{1}{2}\langle X^c\rangle_t$ and $A_t := \prod_{0<s\le t}(1+\Delta X_s)\exp(-\Delta X_s)$. To argue that $A$ is a semimartingale, argue that $X$ has only finitely many "big" jumps (of size $\ge 1/2$, say) on each compact interval, and that for each $t > 0$, the infinite series $\sum_{0<s\le t}\big(\log(1 + \Delta X_s \mathbb{1}_{\{|\Delta X_s|<1/2\}}) - \Delta X_s \mathbb{1}_{\{|\Delta X_s|<1/2\}}\big)$ converges, by using the inequality $|\log(1 + x) - x| \le x^2$ for $|x| < 1/2$. Then apply Itô's formula to $\exp(Y_t)$ and the product formula to $\exp(Y_t)$ and $A_t$. In particular, show that

$$\Delta \exp(Y_t) = \exp(Y_{t-})(\exp(\Delta X_t) - 1) \quad \text{and} \quad \Delta A_t = A_{t-}\exp(-\Delta X_t)(1 + \Delta X_t - \exp(\Delta X_t)).$$

You do not have to argue that the solution to the SDE $\mathrm{d}Z_t = Z_{t-}\,\mathrm{d}X_t$ is unique.

(b) Let $X$ be the compensated compound Poisson process from Exercise 0-1 (c), and assume that $Y_1 > -1$ $\mathbb{P}$-a.s. Show that there exists a *compound Poisson process with drift* $\widetilde{X} = (\widetilde{X}_t)_{t\ge0}$, i.e., $\widetilde{X}_t := \sum_{k=1}^{N_t} \widetilde{Y}_k + \nu t$, where the $\widetilde{Y}_k$ are independent of $N$ and i.i.d., and $\nu \in \mathbb{R}$, such that

$$\mathcal{E}(X) = \exp(\widetilde{X}).$$