ETH Zürich HS 2013
D-MATH
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Mathematical Finance

Exercise Sheet 4

Exercise 4-1

Let $(\Omega, \mathscr{F}, (\mathscr{F}_k)_{k=0,\dots,T}, \mathbb{P})$ be a filtered probability space with $\mathscr{F}_0$ $\mathbb{P}$-trivial and $\bar{S} = (1, S) = (1, S_k^1, \dots, S_k^d)_{k=0,\dots,T}$ a discrete-time model with finite time horizon $T \in \mathbb{N}$. Assume that S satisfies NA. Let $H = (H_k)_{k=0,\dots,T}$ be an American option and $U : [0, \infty) \to [0, \infty)$ an increasing, concave (utility) function. For $k \in \{0, \dots, T\}$, denote by $\mathcal{S}_{k,T}$ the set of all stopping times with values in $\{k, \dots, T\}$. Suppose that the buyer of the American option wants to choose a stopping time $\tau_0^* \in \mathcal{S}_{0,T}$ which is optimal in the sense that it maximises his expected utility $\mathbb{E}[U(H_\tau)]$ from the attained payoff among all stopping times $\tau \in \mathcal{S}_{0,T}$. Assume that $\sup_{\tau \in \mathcal{S}_{0,T}} \mathbb{E}[U(H_\tau)] < \infty$ and define the process $\overline{V} = (\overline{V}_k)_{k=0,\dots,T}$ via backward recursion by

$$\overline{V}_T := U(H_T) \quad \text{and} \quad \overline{V}_k = \max\Big(U(H_k), \mathbb{E}[\overline{V}_{k+1} | \mathscr{F}_k]\Big), \quad k = T-1, \dots, 0.$$

Moreover, for $k \in \{0, \dots, T\}$, define the stopping time

$$\tau_k^* := \inf\{t \in \{k, \dots, T\} : \overline{V}_t = U(H_t)\}.$$

(a) Show that for $k \in \{0, \dots, T\}$, $\overline{V}^{\tau_k^*}$ is a $\mathbb{P}$-martingale on $\{k, \dots, T\}$. Deduce that

$$\mathbb{E}[U(H_{\tau_k^*}) \mid \mathscr{F}_k] = \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_{k,T}} \mathbb{E}[U(H_\tau) \mid \mathscr{F}_k] \;\; \mathbb{P}\text{-a.s.}, \quad k = 0, \dots, T.$$

Hint: Use the identity $\overline{V}_k = \operatorname{ess\,sup}_{\tau \in \mathcal{S}_{k,T}} \mathbb{E}[U(H_\tau) \mid \mathscr{F}_k]$ $\mathbb{P}$-a.s. for $k = 0, \dots, T$ from the lecture.

(b) Suppose now that the market S is complete and that $\mathbb{P}$ is the unique martingale measure for S. Show that there exists a predictable process $\vartheta = (\vartheta_k^1, \dots, \vartheta_k^d)_{k=1,\dots,T}$ such that

$$\overline{V}_0 + \vartheta \bullet S_{\tau_0^*} = U(H_{\tau_0^*}) \;\; \mathbb{P}\text{-a.s.}$$

Exercise 4-2

Let $(\Omega, \mathscr{F}, (\mathscr{F}_k)_{k=0,\dots,T}, \mathbb{P})$ be a filtered probability space with $\mathscr{F}_0$ $\mathbb{P}$-trivial and $\bar{S} = (1, S) = (1, S_k^1, \dots, S_k^d)_{k=0,\dots,T}$ a discrete-time model with time horizon $T \in \mathbb{N}$. Assume that S satisfies NA and that the market S is complete, i.e., there exists a unique equivalent martingale measure $\mathbb{Q} \approx \mathbb{P}$ on $\mathscr{F}_T$ for S. For $k \in \{0, \dots, T\}$, denote by $\mathcal{S}_{k,T}$ the set of all stopping times with values in $\{k, \dots, T\}$. Let $(H_k)_{k=0,\dots,T}$ be an American option, which is uniformly bounded. Assume that the American option is traded at time 0 at a price of $S_0^H \geq 0$. We say that there is a *buyer arbitrage* for H if there exist a predictable process $\vartheta = (\vartheta_k^1, \dots, \vartheta_k^d)_{k=1,\dots,T}$, a constant $c > 0$ and a stopping time $\tau \in \mathcal{S}_{0,T}$ such that

$$\vartheta \bullet S_\tau + c(H_\tau - S_0^H) \geq 0 \ \ \mathbb{P}\text{-a.s.} \quad \text{and} \quad \mathbb{P}[\vartheta \bullet S_\tau + c(H_\tau - S_0^H) > 0] > 0.$$

Similarly, we say that there is a *seller arbitrage* for H if there exist a predictable process $\vartheta = (\vartheta_k^1, \dots, \vartheta_k^d)_{k=1,\dots,T}$ and a constant $c < 0$ such that for *all* stopping times $\tau \in \mathcal{S}_{0,T}$,

$$\vartheta \bullet S_\tau + c(H_\tau - S_0^H) \geq 0 \ \ \mathbb{P}\text{-a.s.} \quad \text{and} \quad \mathbb{P}[\vartheta \bullet S_\tau + c(H_\tau - S_0^H) > 0] > 0.$$

We say that S_0^H is an *arbitrage-free price* for the American option if there exists neither a buyer nor a seller arbitrage for H.

(a) Show that there exists a buyer arbitrage for H if and only if $S_0^H < \sup_{\tau \in \mathcal{S}_{0,T}} \mathbb{E}_{\mathbb{Q}}[H_\tau]$.

Hint: For "$\Leftarrow$", use Exercise 4-1 (b).

(b) Show that there exists a seller arbitrage for H if and only if $S_0^H > \sup_{\tau \in \mathcal{S}_{0,T}} \mathbb{E}_{\mathbb{Q}}[H_\tau]$.

Remark: The above results show that $\sup_{\tau \in \mathcal{S}_{0,T}} \mathbb{E}_{\mathbb{Q}}[H_\tau]$ is the unique arbitrage-free price for H.

Exercise 4-3

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a filtered probability space supporting a Brownian motion $W = (W_t)_{t \geq 0}$. Denote by $(\mathscr{F}_t^W)_{t \geq 0}$ the natural completed filtration of W. Let $\sigma > 0$ and $0 < r < \frac{\sigma^2}{2}$. Consider an *undiscounted* Black-Scholes-type market $(\widetilde{S}^0, \widetilde{S}^1) = (\widetilde{S}_t^0, \widetilde{S}_t^1)_{t \geq 0}$ given by the SDEs

$$\mathrm{d}\widetilde{S}_t^0 = r\widetilde{S}_t^0 \, \mathrm{d}t, \quad \widetilde{S}_0^0 = 1, \quad \text{and} \quad \mathrm{d}\widetilde{S}_t^1 = \widetilde{S}_t^1(r \, \mathrm{d}t + \sigma \, \mathrm{d}W_t), \quad \widetilde{S}_0^1 = s > 0.$$

Denote by $S^1 := \frac{\widetilde{S}^1}{\widetilde{S}^0}$ the discounted stock price. Then $\mathbb{P}$ is the unique equivalent (local) martingale measure for S^1. Denote by $\mathcal{S}_{0,\infty}$ the set of all $\mathbb{P}$-a.s. finite stopping times. The arbitrage-free price of a *perpetual American put option* on $\widetilde{S}^1$ with strike $K > 0$ is given by

$$v(s) := \sup_{\tau \in \mathcal{S}_{0,\infty}} \mathbb{E}\left[\frac{(K - \widetilde{S}_\tau^1)^+}{\widetilde{S}_\tau^0}\right].$$

(a) For $L \in (0, K)$ define the stopping time

$$\tau_L := \inf\{t \geq 0 : \widetilde{S}_t \leq L\}$$

and set

$$v_L(s) = \mathbb{E}\left[\frac{(K - \widetilde{S}_{\tau_L}^1)^+}{\widetilde{S}_{\tau_L}^0}\right].$$

Show that

$$v_L(s) = \begin{cases} K - s, & 0 < s \leq L, \\ (K - L)\left(\frac{s}{L}\right)^{-\frac{2r}{\sigma^2}}, & s > L. \end{cases}$$

Hint: Use without proof that the stopping time $\sigma_{a,b} := \inf\{t \geq 0 : W_t \leq -a + bt\}$, where $a, b > 0$, has the Laplace transform

$$\mathbb{E}[\exp(-\lambda \sigma_{a,b})] = \exp\left(-a(\sqrt{b^2 + 2\lambda} - b)\right), \quad \lambda \geq 0.$$

(b) Show that there exists a unique $L^* \in (0, K)$ such that $v_{L^*}(s) \geq v_L(s)$ for all $L \in (0, K)$ and all $s \in (0, \infty)$. In particular, show that $v_{L^*}(s) \geq (K - s)^+$ for all $s \in (0, \infty)$.

Hint: Define the function $g : (0, K) \to (0, \infty)$ by $g(L) := (K - L)L^{\frac{2r}{\sigma^2}}$, and show that it has a unique global maximum L^* on $(0, K)$. In addition, for $L \in (0, K)$, define the function $h_L : (0, \infty) \to (0, \infty)$ by $h_L(s) = s^{-\frac{2r}{\sigma^2}} g(L)$, show that $h_{L^*}(L^*) = K - L^*$ and $h'_{L^*}(L^*) = -1$, and use that h_L is strictly convex for all $L \in (0, K)$.

(c) Show that the process $\widetilde{V} = (\widetilde{V}_t)_{t \geq 0}$ defined by $\widetilde{V}_t := \exp(-rt) v_{L^*}(\widetilde{S}_t)$, $t \geq 0$, is a $\mathbb{P}$-supermartingale. Deduce that τ_{L^*} satisfies

$$\mathbb{E}\left[\frac{(K - \widetilde{S}^1_{\tau_{L^*}})^+}{\widetilde{S}^0_{\tau_{L^*}}}\right] = \sup_{\tau \in \mathcal{S}_{0,\infty}} \mathbb{E}\left[\frac{(K - \widetilde{S}^1_\tau)^+}{\widetilde{S}^0_\tau}\right].$$

Hint: First, show that v_{L^*} is in $C^1((0, \infty)) \cap C^2((0, \infty) \setminus \{L^*\})$ and satisfies

$$-r v_{L^*}(s) + r s v'_{L^*}(s) + \frac{1}{2}\sigma^2 s^2 v''_{L^*}(s) \leq 0, \quad s \in (0, \infty) \setminus \{L^*\}.$$

Next, use without proof that if S is a strictly positive semimartingale and $f : (0, \infty) \to \mathbb{R}$ is in $C^1((0, \infty))$ and there exists a *finite* set $A \subset (0, \infty)$ such that $f \in C^2((0, \infty) \setminus A)$ and $f'' \mathbb{1}_{\{f \notin A\}}$ is *bounded* on compact sets, then $f(S)$ is again a semimartingale and Itô's formula holds with f'' replaced by $f'' \mathbb{1}_{\{f \notin A\}}$.

Exercise 4-4

Let $(\Omega, \mathscr{F}, (\mathscr{F}_k)_{k=0,1}, \mathbb{P})$ be a filtered probability space and $\bar{S} = (1, S^1_k, \ldots, S^d_k)_{k=0,1}$ a one-period model. Assume that $\mathscr{F}_0 = \{\emptyset, \Omega\}$ and that S is *non-redundant* in the sense that for each $\vartheta \in \mathbb{R}^d$, we have $\vartheta^{tr} \Delta S_1 = 0$ $\mathbb{P}$-a.s. if and only if $\vartheta = 0$. Let $U : (0, \infty) \to \mathbb{R}$ be a utility function (without Inada conditions), i.e., U is strictly increasing, strictly concave and in C^1. Set $U(0) := \lim_{t \downarrow\downarrow 0} U(t) \in [-\infty, \infty)$ and $U(\infty) := \lim_{t \uparrow\uparrow \infty} U(t) \in (-\infty, +\infty]$. For $x \geq 0$, set

$$\mathcal{A}(x) := \{\vartheta \in \mathbb{R}^d : x + \vartheta^{tr} \Delta S_1 \geq 0 \ \mathbb{P}\text{-a.s.}\},$$
$$u(x) := \sup_{\vartheta \in \mathcal{A}(x)} \mathbb{E}[U(x + \vartheta^{tr} \Delta S_1)],$$

where $\mathbb{E}[U(x + \vartheta^{tr} \Delta S_1)] := -\infty$ if $U(x + \vartheta^{tr} \Delta S_1)^- \notin L^1(\mathbb{P})$.

(a) Fix $x \geq 0$. Show that the set $\mathcal{A}(x)$ is compact if and only if S satisfies NA.

Hint: For "$\Leftarrow$", argue by contradiction and assume that there exists a sequence $(\vartheta_n)_{n \in \mathbb{N}}$ in $\mathcal{A}(x) \setminus \{0\}$ such that $\lim_{n \to \infty} \|\vartheta_n\|_\infty = +\infty$. For $n \in \mathbb{N}$, set $\eta_n := \frac{\vartheta_n}{\|\vartheta_n\|_\infty}$, consider the sequence $(\eta_n)_{n \in \mathbb{N}}$ and use non-redundancy of S.

(b) Suppose that $S_1^i \in L^1(\mathbb{P})$ for $i \in \{1, \ldots, d\}$ and $U(\infty) = +\infty$. Fix $x > 0$. Show that $u(x) < \infty$ if and only if S satisfies NA.

Hint: For "$\Leftarrow$", construct $Y \in L^1(\mathbb{P})$ such that $U(x + \vartheta^{tr} \Delta S_1) \leq Y$ $\mathbb{P}$-a.s. for all $\vartheta \in \mathcal{A}(x)$ using concavity of U and part (a).

(c) Suppose that $S_1^i \in L^1(\mathbb{P})$ for $i \in \{1, \ldots, d\}$ and that S satisfies NA. Fix $x > 0$. Show that there is a unique $\vartheta^* \in \mathcal{A}(x)$ such that

$$\mathbb{E}[U(x + (\vartheta^*)^{tr} \Delta S_1)] = u(x) < \infty.$$

Hint: Use parts (a) and (b) and Fatou's lemma. Moreover, use without proof that U is strictly concave on $[0, \infty)$ in case that $U(0) > -\infty$.

Exercise 4-5

Consider the same setup and notation as in Exercise 4-4. Assume that $U(0) > -\infty$, that S satisfies NA and that $S_1^i \in L^1(\mathbb{P})$ for $i \in \{1, \ldots, d\}$. Fix $x > 0$ and assume that the unique $\vartheta^* \in \mathcal{A}(x)$ satisfying $\mathbb{E}[U(x + (\vartheta^*)^{tr} \Delta S_1)] = u(x) < \infty$ is in the interior of $\mathcal{A}(x)$.

(a) Fix $z \geq 0$. Using only the concavity property, show that the function

$$y \mapsto \frac{U(y) - U(z)}{y - z}, \quad y \in (0, \infty) \setminus \{z\},$$

is decreasing.

Remark: This shows in particular that $U'(0) := \lim_{h \downarrow\downarrow 0} \frac{U(h) - U(0)}{h} \in (0, +\infty]$ is well defined.

(b) Show that $U'\left(x + (\vartheta^*)^{tr} \Delta S_1\right) < \infty$ $\mathbb{P}$-a.s., that

$$U'\left(x + (\vartheta^*)^{tr} \Delta S_1\right) \Delta S_1^i \in L^1(\mathbb{P}), \quad i \in \{1, \ldots, d\},$$

and derive the *first-order condition*

$$\mathbb{E}[U'(x + (\vartheta^*)^{tr} \Delta S_1) \Delta S_1^i] = 0, \quad i \in \{1, \ldots, d\}.$$

Hint: Let $\eta \in \mathbb{R}^d \setminus \{0\}$, and consider the limit

$$\lim_{\epsilon \downarrow\downarrow 0} \frac{U(x + (\vartheta^* + \epsilon\eta)^{tr} \Delta S_1) - U(x + (\vartheta^*)^{tr} \Delta S_1)}{\epsilon}$$

using part (a).

(c) Show that there exists an equivalent martingale measure $\mathbb{Q} \approx \mathbb{P}$ on $\mathscr{F}_1$ for S with density

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \frac{U'(x + (\vartheta^*)^{tr} \Delta S_1)}{\mathbb{E}[U'(x + (\vartheta^*)^{tr} \Delta S_1)]}.$$

Remark: The above result is a constructive proof of the Dalang-Morton-Willinger theorem in our setup.