ETH Zürich HS 2013
D-MATH
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Mathematical Finance

Exercise Sheet 3

Solution 3-1

(a) "$\Rightarrow$": This is trivial, as every martingale is by definition integrable.

"$\Leftarrow$": Let $(\tau_n)_{n\in\mathbb{N}}$ be a localising sequence for X. First, we show by backward induction that

$$X_k^- \in L^1(\mathbb{P}) \text{ for all } k = T, \dots, 0.$$

The induction basis is trivial. For the induction step, let $1 \le k \le T$ and suppose that $X_k^- \in L^1(\mathbb{P})$. Fix $n \in \mathbb{N}$, Then $(X^{\tau_n})^-$ is a submartingale as the function $x \mapsto x^-$ is convex and $\mathbb{E}[(X_k^{\tau_n})^-] \le \mathbb{E}[|X_k^{\tau_n}|] < \infty$ since X^{τ_n} is a martingale and hence integrable. The submartingale property yields

$$\begin{aligned} X_{k-1}^- \mathbb{1}_{\{\tau_n > k-1\}} &= (X_{k-1}^{\tau_n})^- \mathbb{1}_{\{\tau_n > k-1\}} \le \mathbb{E}[(X_k^{\tau_n})^- \mid \mathscr{F}_{k-1}] \mathbb{1}_{\{\tau_n > k-1\}} \\ &= \mathbb{E}[(X_k^{\tau_n})^- \mathbb{1}_{\{\tau_n > k-1\}} \mid \mathscr{F}_{k-1}] = \mathbb{E}[X_k^- \mathbb{1}_{\{\tau_n > k-1\}} \mid \mathscr{F}_{k-1}] \\ &= \mathbb{E}[X_k^- \mid \mathscr{F}_{k-1}] \mathbb{1}_{\{\tau_n > k-1\}} \ \mathbb{P}\text{-a.s.} \end{aligned} \tag{1}$$

Letting $n \to \infty$ shows that $X_{k-1}^- \le \mathbb{E}[X_k^- \mid \mathscr{F}_{k-1}]$ and taking expectations yields $X_{k-1}^- \in L^1(\mathbb{P})$. Next, we show that also X is integrable. To this end fix $0 \le k \le T$. Since X^- is integrable, the expectation $\mathbb{E}[X_k]$ is well-defined (it may be $+\infty$). Using that $X_k^{\tau_n} \ge -\sum_{j=0}^T (X_j)^- \in L^1(\mathbb{P})$, we may apply Fatou's lemma and get

$$\mathbb{E}[X_k] \le \liminf_{n\to\infty} \mathbb{E}[X_k^{\tau_n}] = \mathbb{E}[X_0] = 0.$$

Finally, we show that X is a martingale. The integrability of X implies the integrability of the maximum process since

$$\max_{j\in\{0,\dots,k\}} |X_j| \le \sum_{\ell=0}^k |X_\ell| \in L^1(\mathbb{P}) \quad \text{for } k = 0, \dots, T. \tag{2}$$

Thus, by dominated convergence

$$\mathbb{E}\left[X_{k+1} \,\middle|\, \mathscr{F}_k\right] = \lim_{n\to\infty} \mathbb{E}\left[X_{k+1}^{\tau_n} \,\middle|\, \mathscr{F}_k\right] = \lim_{n\to\infty} X_k^{\tau_n} = X_k \quad \mathbb{P}\text{-a.s.} \tag{3}$$

(b) First, suppose that X is a local martingale. Let $(\tau_n)_{n\in\mathbb{N}}$ be a localising sequence for X. Let ϑ be a predictable process such that $(\vartheta \bullet X)_T^- \in L^1(\mathbb{P})$. For $n \in \mathbb{N}$, define the stopping time

$$\sigma_n := \inf\{k \ge 0 : |\vartheta_{k+1}| \ge n\}. \tag{4}$$

Note that this is indeed a stopping time because ϑ is predictable. Then $(\sigma_n)_{n\in\mathbb{N}}$ is increasing to $+\infty$ $\mathbb{P}$-a.s. For $n \in \mathbb{N}$, define the stopping time $\rho_n := \tau_n \wedge \sigma_n$. Then $(\rho_n)_{n\in\mathbb{N}}$ is increasing to $+\infty$ $\mathbb{P}$-a.s. Moreover, for each $n \in \mathbb{N}$, X^{ρ_n} is a martingale and $|\vartheta_k| \mathbb{1}_{\{k \le \rho_n\}} \le n$,

$k \in \{0, \dots, T\}$. Therefore, for each $n \in \mathbb{N}$, $(\vartheta \bullet X)^{\rho_n}$ is a martingale null at 0. Indeed, for $k \in \{1, \dots, T\}$,

$$\mathbb{E}[|(\vartheta \bullet X)^{\rho_n}_k|] = \mathbb{E}\Big[\Big|\sum_{j=1}^{k} \vartheta_j \mathbb{1}_{\{j \leq \rho_n\}} \Delta X^{\rho_n}_k\Big|\Big] \leq n \sum_{j=1}^{k} \mathbb{E}[|\Delta X^{\rho_n}_j|] < \infty,$$

$$\begin{aligned}\mathbb{E}[(\vartheta \bullet X)^{\rho_n}_k - (\vartheta \bullet X)^{\rho_n}_{k-1} \mid \mathscr{F}_{k-1}] &= \mathbb{E}[\vartheta_k \Delta X^{\rho_n}_k \mid \mathscr{F}_{k-1}] \\ &= \vartheta_k \mathbb{E}[\Delta X^{\rho_n}_k \mid \mathscr{F}_{k-1}] = 0 \ \mathbb{P}\text{-a.s.}\end{aligned} \tag{5}$$

Thus $\vartheta \bullet X$ is a local martingale with $(\vartheta \bullet X)^-_T \in L^1(\mathbb{P})$. By part (a) it is even a true martingale and thus $\vartheta \bullet X_T \in L^1(\mathbb{P})$ and $\mathbb{E}[\vartheta \bullet X_T] = 0$.

Conversely, assume the stated condition. Let $(\tau_n)_{n\in\mathbb{N}}$ be a sequence of stopping times, which is $\mathbb{P}$-a.s. increasing to $+\infty$, such that X^{τ_n} is integrable for all $n \in \mathbb{N}$. We proceed to show that for each $n \in \mathbb{N}$, X^{τ_n} is a martingale, and so X is a local martingale. To this end, let $k \in \{0, \dots, T-1\}$ and $A \in \mathscr{F}_k$ be arbitrary. Define the process $\vartheta = (\vartheta_j)_{j=0,\dots,T}$ by

$$\vartheta_j := \begin{cases} \mathbb{1}_{A\cap\{k+1\leq\tau_n\}}, & \text{if } j = k+1, \\ 0, & \text{else.} \end{cases} \tag{6}$$

Since τ_n is a stopping time, $A \cap \{k+1 \leq \tau_n\} \in \mathscr{F}_k$, and hence ϑ is predictable. Next, note that

$$\vartheta \bullet X_T = \mathbb{1}_{A\cap\{k+1\leq\tau_n\}} \Delta X_{k+1} = \mathbb{1}_A \Delta X^{\tau_n}_{k+1},$$

This implies in particular that $\vartheta \bullet X_T$ is integrable, and hence by assumption,

$$\mathbb{E}[\mathbb{1}_A \Delta X^{\tau_n}_{k+1}] \leq 0 \tag{7}$$

The same argument with $-\vartheta$ instead of ϑ, show that

$$\mathbb{E}[-\mathbb{1}_A \Delta X^{\tau_n}_{k+1}] \leq 0. \tag{8}$$

and thus we may conclude that $\mathbb{E}[\mathbb{1}_A \Delta X^{\tau_n}_{k+1}] = 0$ Since $A \in \mathscr{F}_k$ was arbitrary, this implies that $\mathbb{E}[\Delta X^{\tau_n}_{k+1} \mid \mathscr{F}_k] = 0$. Since $k \in \{0, \dots, T-1\}$ was arbitrary, we conclude that X^{τ_n} is a martingale.

Solution 3-2

(a) For $n \in \mathbb{N}$, define the stopping time $\tau_n := \inf\{t > 0 : X_t < 1/n\}$. Then by right-continuity of X, $X_{\tau_n} \leq 1/n$ on $\{\tau_n < \infty\}$ for $n \in \mathbb{N}$. Hence, by the optional stopping theorem, for all $n \in \mathbb{N}$,

$$\mathbb{E}[X_t \mathbb{1}_{\{\tau_n \leq t\}}] \leq \mathbb{E}[X_{\tau_n} \mathbb{1}_{\{\tau_n \leq t\}}] \leq 1/n, \quad t \geq 0. \tag{9}$$

Since $\tau_0 = \lim_{n\to\infty} \tau_n$ $\mathbb{P}$-a.s., nonnegativity of X and dominated convergence give

$$\mathbb{E}[X_t \mathbb{1}_{\{\tau_0 \leq t\}}] = 0, \quad t \geq 0. \tag{10}$$

This implies that $X_t = 0$ on $\{\tau_0 \leq t\}$ $\mathbb{P}$-a.s. for each $t \geq 0$, and right-continuity of X establishes the claim.

(b) First, note that since X is a strictly positive local martingale, it is a strictly positive supermartingale by Fatou's lemma and hence $X_- > 0$ $\mathbb{P}$-a.s. by part (a). This implies that the process $\frac{1}{X_-}$ is well-defined. Since it is adapted and left-continuous, it is in addition predictable and locally bounded. Hence by the hint, the process $M = (M_t)_{t\geq0}$ defined by

$$M_t := \int_0^t \frac{1}{X_{s-}} \,\mathrm{d}X_s, \quad t \geq 0, \tag{11}$$

is well defined and a local martingale. Moreover, associativity of the stochastic integral gives

$$\int_0^t X_{s-}\,\mathrm{d}M_s = \int_0^t \frac{X_{s-}}{X_{s-}}\,\mathrm{d}X_s = X_t - X_0 = X_t - 1, \quad t \geq 0. \tag{12}$$

This shows existence of M.

To establish uniqueness, suppose that $\widetilde{M}$ is a local martingale null at 0 such that $X = \mathcal{E}(\widetilde{M})$. Then associativity of the stochastic integral together with the definition of the stochastic exponential give

$$\widetilde{M}_t = \int_0^t \frac{1}{X_{s-}} X_{s-}\,\mathrm{d}M_s = \int_0^t \frac{1}{X_{s-}}\,\mathrm{d}X_s = M_t, \quad t \geq 0. \tag{13}$$

Solution 3-3

(a) Recall that we can write $N = N^c + N^d + N^{FV}$, where $N^c \in \mathcal{H}^{2,c}_{0,\mathrm{loc}}$, $N^d \in \mathcal{H}^{2,d}_{0,\mathrm{loc}}$ and N^{FV} is a local martingale of finite variation and null at 0. (More precisely, as N is a semimartingale we can write $N = N^1 + N^2$, where $N^1 \in \mathcal{H}^2_{0,\mathrm{loc}}$ and N^2 is adapted, of finite variation and null at 0. Since both N and N^1 are local martingales, the same is true for $N^{FV} := N^2$. Decomposing $N^1 = N^c + N^d$, where $N^c \in \mathcal{H}^{2,c}_{0,\mathrm{loc}}$, $N^d \in \mathcal{H}^{2,d}_{0,\mathrm{loc}}$, establishes the claim.)

Note that since $M \in \mathcal{H}^{2,c}_{0,\mathrm{loc}}$ and $(N^{FV})^c = 0$,

$$[M, N^d] \equiv 0 \quad \text{and} \quad [M, N^{FV}] = \sum \Delta M \Delta N^{FV} \equiv 0. \tag{14}$$

Now applying the usual Kunita-Watanabe decomposition to N^c, we get $H \in L^2_{\mathrm{loc}}(M)$ and $L^c \in \mathcal{H}^{2,c}_{0,\mathrm{loc}}$ such that $N^c = H \bullet M + L^c$ and $[M, L^c] = \langle M, L^c \rangle \equiv 0$. Now set $L := L^c + N^d + N^{FV}$. Then L is a local martingale, $N = H \bullet M + L$ and

$$[M, L] = [M, L^c] + [M, N^d] + [M, N^{FV}] \equiv 0. \tag{15}$$

(b) First, assume that S satisfies SC, and let $H \in L^2_{\mathrm{loc}}(M)$ be such that $A = \int H\,\mathrm{d}\langle M \rangle$. Then $-H \bullet M$ is a continuous local martingale null at 0. Set $Z := \mathcal{E}(-H \bullet M)$. Then Z is a strictly positive continuous local martingale with $Z_0 = 1$. We show that Z is an equivalent local martingale deflator. By the product rule and the structure condition,

$$\begin{aligned}
\mathrm{d}(Z_t S_t) &= S_t\,\mathrm{d}Z_t + Z_t\,\mathrm{d}S_t + \mathrm{d}\langle Z, S \rangle_t \\
&= S_t\,\mathrm{d}Z_t + Z_t\,\mathrm{d}M_t + Z_t\,\mathrm{d}A_t - Z_t H_t\,\mathrm{d}\langle M, M \rangle_t \\
&= S_t\,\mathrm{d}Z_t + Z_t\,\mathrm{d}M_t.
\end{aligned} \tag{16}$$

Since Z_t and M are continuous local martingales, $\int S\,\mathrm{d}Z$ and $\int Z\,\mathrm{d}M$ are so, too, and this establishes the claim. Conversely, assume that there exists an equivalent local martingale deflator Z for S. The by Exercise 3-2 (b), we can write $Z = \mathcal{E}(N)$, where $N = (N_t)_{t \geq 0}$ is a local martingale null at 0. By part (a), we may write – using a change of sign for convenience – $N = -H \bullet M + L$, where $H \in L^2_{\mathrm{loc}}(M)$ and $L = (L_t)_{t \geq 0}$ is a local martingale null at 0 and such that $[M, L] \equiv 0$. Then by the product rule and using that $[M, L] \equiv 0$,

$$\begin{aligned}
\mathrm{d}(Z_t S_t) &= S_{t-}\,\mathrm{d}Z_t + Z_{t-}\,\mathrm{d}S_t + \mathrm{d}[Z, S]_t \\
&= S_{t-}\,\mathrm{d}Z_t + Z_{t-}\,\mathrm{d}M_t + Z_{t-}\,\mathrm{d}A_t - Z_{t-} H_t\,\mathrm{d}[M, M]_t + Z_{t-}\,\mathrm{d}[M, L]_t \\
&= S_{t-}\,\mathrm{d}Z_t + Z_{t-}\,\mathrm{d}M_t + Z_{t-}\,\mathrm{d}A_t - Z_{t-} H_t\,\mathrm{d}\langle M, M \rangle_t.
\end{aligned} \tag{17}$$

Since ZS is a local martingale by hypothesis and $\int S_-\,\mathrm{d}Z$ and $\int Z_-\,\mathrm{d}M$ are local martingales as integrals of a locally bounded process against a local martingale, it follows that

$\int Z_- \, \mathrm{d}A - \int Z_- H \, \mathrm{d}\langle M, M \rangle$ is a local martingale, too. As it is continuous, of finite variation and null at 0, it is 0 identically. Since $1/Z_-$ is predictable and locally bounded, associativity of the stochastic integral gives

$$A_t = \int_0^t \frac{1}{Z_{s-}} Z_{s-} \, \mathrm{d}A_s = \int_0^t \frac{1}{Z_{s-}} Z_{s-} H_s \, \mathrm{d}\langle M, M \rangle_s = \int_0^t H_s \, \mathrm{d}\langle M, M \rangle_s. \tag{18}$$

This shows that S satisfies SC.

Solution 3-4

(a) Define the process $R = (R_t)_{t \in [0,T]}$ by

$$\begin{aligned} R_t &:= \mu t + \frac{\sigma}{\sqrt{\lambda}} \widetilde{N}_t = \mu t + \frac{\sigma}{\sqrt{\lambda}} (N_t - \lambda t) = (\mu - \sigma \sqrt{\lambda}) t + \frac{\sigma}{\sqrt{\lambda}} N_t \\ &= \frac{\sigma}{\sqrt{\lambda}} (N_t - \ell t), \quad t \in [0,T], \end{aligned} \tag{19}$$

where $\ell := \lambda - \frac{\mu}{\sigma} \sqrt{\lambda}$. It follows from Exercise 1-5 (b) that S fails NA, and a fortiori NFLVR, if the paths of R are monotone, i.e., if $\ell \leq 0$. On the other hand, if $\ell > 0$, define the measure $\mathbb{Q}^\lambda \approx \mathbb{P}$ on $\mathscr{F}_T$ by

$$\frac{\mathrm{d}\mathbb{Q}^\lambda}{\mathrm{d}\mathbb{P}} = \exp\left(\sum_{k=1}^{N_T} \log \frac{\ell}{\lambda} + (\lambda - \ell) T \right). \tag{20}$$

Then it follows from Exercise 1-4 that under $\mathbb{Q}^\lambda$, $R = \frac{\sigma}{\sqrt{\lambda}} \widetilde{N}^{\mathbb{Q}^\lambda}$, where $N^{\mathbb{Q}^\lambda} := N$ is a Poisson process with rate ℓ. Since R is a $\mathbb{Q}^\lambda$-martingale, it follows from Exercise 1-5 (a) that S is so, too.

(b) Since S admits a unique equivalent martingale measure $\mathbb{Q}^\lambda$, the arbitrage-free price of $\mathbb{1}_{\{S_T > K\}}$ is given by

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}^\lambda}[\mathbb{1}_{\{S_T > K\}}] &= \mathbb{Q}^\lambda[S_T > K] \\ &= \mathbb{Q}^\lambda \left[S_0 \exp\left(\log\left(1 + \frac{\sigma}{\sqrt{\lambda}}\right) N_T^{\mathbb{Q}^\lambda} - \frac{\sigma \ell}{\sqrt{\lambda}} T \right) > K \right] \\ &= \mathbb{Q}^\lambda \left[N_T^{\mathbb{Q}^\lambda} > \frac{\log \frac{K}{S_0} + \frac{\sigma \ell}{\sqrt{\lambda}} T}{\log\left(1 + \frac{\sigma}{\sqrt{\lambda}}\right)} \right] \\ &= \overline{\Psi}_{(\lambda - \frac{\mu}{\sigma}\sqrt{\lambda})T} \left(\frac{\log \frac{K}{S_0} + \left(\sigma\sqrt{\lambda} - \mu\right) T}{\log\left(1 + \frac{\sigma}{\sqrt{\lambda}}\right)} \right). \end{aligned} \tag{21}$$

(c) First, define $\widetilde{\mathbb{Q}}^\lambda \approx \mathbb{Q}^\lambda$ on $\mathscr{F}_T$ by $\frac{\mathrm{d}\widetilde{\mathbb{Q}}^\lambda}{\mathrm{d}\mathbb{Q}^\lambda} := S_T/S_0$. Note that

$$S_T/S_0 = \mathcal{E}(R)_T = \exp\left(\sum_{k=1}^{N_T^{\mathbb{Q}^\lambda}} \log \frac{\widetilde{\ell}}{\ell} + (\ell - \widetilde{\ell}) T \right), \tag{22}$$

where $\widetilde{\ell} := \left(1 + \frac{\sigma}{\sqrt{\lambda}}\right) \ell$. Now it follows from Exercise 1-4 that under $\widetilde{\mathbb{Q}}^\lambda$,

$$R_t = \frac{\sigma}{\sqrt{\lambda}} N_t^{\widetilde{\mathbb{Q}}^\lambda} - \frac{\sigma}{\sqrt{\lambda}} \ell t, \quad t \in [0,T], \tag{23}$$

where $N^{\widetilde{\mathbb{Q}}^\lambda}$ is a Poisson process with rate $\widetilde{\ell}$.

Next, since S admits a unique equivalent martingale measure $\mathbb{Q}^\lambda$, the arbitrage-free price of $S_T \mathbb{1}_{\{S_T > K\}}$ is given by $\mathbb{E}_{\mathbb{Q}^\lambda}[S_T \mathbb{1}_{\{S_T>K\}}]$. By Bayes' formula and the above and noting that under $\widetilde{\mathbb{Q}}^\lambda$, the calculation is exactly the same as in part (b),

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}^\lambda}[S_T \mathbb{1}_{\{S_T>K\}}] = \mathbb{E}_{\widetilde{\mathbb{Q}}^\lambda}[S_0 \mathbb{1}_{\{S_T>K\}}] &= S_0 \widetilde{\mathbb{Q}}^\lambda[S_T > K] \\ &= S_0 \overline{\Psi}_{\left(1+\frac{\sigma}{\sqrt{\lambda}}\right)(\lambda - \frac{\mu}{\sigma}\sqrt{\lambda})T} \left(\frac{\log \frac{K}{S_0} + \left(\sigma\sqrt{\lambda} - \mu\right) T}{\log\left(1 + \frac{\sigma}{\sqrt{\lambda}}\right)} \right). \qquad (24)\end{aligned}$$

(d) First, it follows immediately from parts (b) and (c) that

$$\begin{aligned}C_0^\lambda &= \mathbb{E}_{\mathbb{Q}^\lambda}[(S_T - K)^+] = \mathbb{E}_{\mathbb{Q}^\lambda}[S_T \mathbb{1}_{\{S_T>K\}}] - K\mathbb{E}_{\mathbb{Q}^\lambda}[\mathbb{1}_{\{S_T>K\}}] \\ &= S_0 \overline{\Psi}_{\left(1+\frac{\sigma}{\sqrt{\lambda}}\right)(\lambda - \frac{\mu}{\sigma}\sqrt{\lambda})T} \left(\frac{\log \frac{K}{S_0} + \left(\sigma\sqrt{\lambda} - \mu\right) T}{\log\left(1 + \frac{\sigma}{\sqrt{\lambda}}\right)} \right) \\ &\quad - K \overline{\Psi}_{(\lambda - \frac{\mu}{\sigma}\sqrt{\lambda})T} \left(\frac{\log \frac{K}{S_0} + \left(\sigma\sqrt{\lambda} - \mu\right) T}{\log\left(1 + \frac{\sigma}{\sqrt{\lambda}}\right)} \right). \qquad (25)\end{aligned}$$

Next, for $\rho > 0$, let F_ρ be the distribution function of $\frac{X_\rho - \rho}{\sqrt{\rho}}$, where X_ρ is Poisson distributed with parameter ρ. Moreover, set $\overline{F}_\rho := 1 - F_\rho$ and $\overline{\Phi} = 1 - \Phi$. Then by the hint, F_ρ converges pointwise to Φ as $\rho \to \infty$, and the convergence is even uniform as Φ is continuous. Thus $\overline{F}_\rho$ converges uniformly to $\overline{\Phi}$ as $\rho \to \infty$. Now the claim follows from the fact that $\overline{\Psi}_\rho(x) = \overline{F}_\rho\left(\frac{x-\rho}{\sqrt{\rho}}\right)$, the fact that $\overline{\Phi}(x) = \Phi(-x)$ and the limits

$$\begin{aligned}&\lim_{\lambda\to\infty} \log\left(1 + \frac{\sigma}{\sqrt{\lambda}}\right) \sqrt{\left(\lambda - \frac{\mu}{\sigma}\sqrt{\lambda}\right) T} = \sigma\sqrt{T}, \\ &\lim_{\lambda\to\infty} \log\left(1 + \frac{\sigma}{\sqrt{\lambda}}\right) \sqrt{\left(1 + \frac{\sigma}{\sqrt{\lambda}}\right)\left(\lambda - \frac{\mu}{\sigma}\sqrt{\lambda}\right) T} = \sigma\sqrt{T}, \\ &\lim_{\lambda\to\infty} \left(\left(\sigma\sqrt{\lambda} - \mu\right) T - \log\left(1 + \frac{\sigma}{\sqrt{\lambda}}\right)\left(\lambda - \frac{\mu}{\sigma}\sqrt{\lambda}\right) T \right) = \frac{\sigma^2}{2} T, \\ &\lim_{\lambda\to\infty} \left(\left(\sigma\sqrt{\lambda} - \mu\right) T - \log\left(1 + \frac{\sigma}{\sqrt{\lambda}}\right)\left(1 + \frac{\sigma}{\sqrt{\lambda}}\right)\left(\lambda - \frac{\mu}{\sigma}\sqrt{\lambda}\right) T \right) = -\frac{\sigma^2}{2} T, \qquad (26)\end{aligned}$$

where we have used that

$$\begin{aligned}\log\left(1 + \frac{\sigma}{\sqrt{\lambda}}\right) &= \frac{\sigma}{\sqrt{\lambda}} - \frac{\sigma^2}{2\lambda} + O\left(\frac{1}{\lambda^{3/2}}\right), \\ \sqrt{\lambda - \frac{\mu}{\sigma}\sqrt{\lambda}} &= \sqrt{\lambda}\sqrt{1 + O\left(\frac{1}{\sqrt{\lambda}}\right)}, \\ \sqrt{\left(1 + \frac{\sigma}{\sqrt{\lambda}}\right)\left(\lambda - \frac{\mu}{\sigma}\sqrt{\lambda}\right)} &= \sqrt{\lambda}\sqrt{1 + O\left(\frac{1}{\sqrt{\lambda}}\right)}. \qquad (27)\end{aligned}$$

Remark: More generally, one can show that if N^λ is a Poisson process with rate λ, then the normalised compensated Poisson process $\frac{\widetilde{N}^\lambda}{\sqrt{\lambda}}$ converges weakly to a standard Brownian motion. But this is of course much more difficult.

Solution 3-5

(a) The idea is to change the jump intensity of N to ℓ using Exercise 1-4 and then the drift of R to $-a\ell$ using Girsanov's theorem. To this end, note that

$$R_t = a(N_t - \ell t) + \sigma\left(W_t + \frac{\mu + a\ell}{\sigma}t\right). \tag{28}$$

Define the measure $\mathbb{P}^\ell \approx \mathbb{P}$ on $\mathscr{F}_T$ by

$$\frac{\mathrm{d}\mathbb{P}^\ell}{\mathrm{d}\mathbb{P}} := \exp\left(\sum_{k=1}^{N_T} \log\frac{\ell}{\lambda} + (\lambda - \ell)T\right). \tag{29}$$

Then by Exercise 1-4, $N^{\mathbb{P}^\ell} := N$ is a Poisson process with rate ℓ under $\mathbb{P}^\ell$. Moreover, as $\frac{\mathrm{d}\mathbb{P}^\ell}{\mathrm{d}\mathbb{P}}$ is a functional of N and N and W are independent under $\mathbb{P}$, it follows that W is a Brownian motion and independent from $N^{\mathbb{P}^\ell}$ under $\mathbb{P}^\ell$, too. Next, define the measure $\mathbb{Q}^\ell \approx \mathbb{P}^\ell \approx \mathbb{P}$ on $\mathscr{F}_T$ by

$$\frac{\mathrm{d}\mathbb{Q}^\ell}{\mathbb{P}^\ell} := \mathcal{E}\left(-\frac{\mu + a\ell}{\sigma}W\right)_T. \tag{30}$$

Then by Girsanov's theorem, $W_t^{\mathbb{Q}^\ell} := W_t + \frac{\mu + a\ell}{\sigma}t$ is a Brownian motion under $\mathbb{Q}^\ell$. Moreover, as $\frac{\mathrm{d}\mathbb{Q}^\ell}{\mathrm{d}\mathbb{P}^\ell}$ is a functional of W, and $N^{\mathbb{P}^\ell}$ and W are independent under $\mathbb{P}^\ell$, it follows that $W^{\mathbb{Q}^\ell}$ is a Brownian motion and independent from $N^{\mathbb{Q}^\ell} := N^{\mathbb{P}^\ell}$ under $\mathbb{Q}^\ell$, too.

(b) It suffices to show that

$$\limsup_{\ell\to\infty} \mathbb{E}_{\mathbb{Q}^\ell}[\mathbb{1}_{\{S_T > K\}}] = 0. \tag{31}$$

First, fix $\ell > 0$. Then by Exercise 1-4 and independence of $W^{\mathbb{Q}^\ell}$ and $N^{\mathbb{Q}^\ell}$,

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}^\ell}[\mathbb{1}_{\{S_T > K\}}] \\
&= \mathbb{Q}^\ell\left[\exp\left(\sigma W_T^{\mathbb{Q}^\ell} - \frac{\sigma^2}{2}T + \log(1+a)N_T^{\mathbb{Q}^\ell} - a\ell T\right) > \frac{K}{S_0}\right] \\
&= \mathbb{Q}^\ell\left[N_T^{\mathbb{Q}^\ell} - \ell T > \frac{\log\frac{K}{S_0} - \sigma W_T^{\mathbb{Q}^\ell} + \frac{\sigma^2}{2}T + (a - \log(1+a))\ell T}{\log(1+a)}\right] \\
&\leq \mathbb{E}_{\mathbb{Q}^\ell}\left[\mathbb{Q}^\ell\left[|N_T^{\mathbb{Q}^\ell} - \ell T| > \frac{\log\frac{K}{S_0} - \sigma w + \frac{\sigma^2}{2}T + (a - \log(1+a))\ell T}{\log(1+a)}\right]\Bigg|_{w = W_T^{\mathbb{Q}^\ell}}\right] \\
&= \int_{\mathbb{R}} \mathbb{Q}^\ell\left[|N_T^{\mathbb{Q}^\ell} - \ell T| > \frac{\log\frac{K}{S_0} - \sigma w + \frac{\sigma^2}{2}T + (a - \log(1+a))\ell T}{\log(1+a)}\right]\frac{\exp(-\frac{w^2}{2T})}{\sqrt{2T\pi}}\,\mathrm{d}w.
\end{aligned} \tag{32}$$

Now, for fixed $w \in \mathbb{R}$, $\log\frac{K}{S_0} - \sigma w + \frac{\sigma^2}{2}T + (a - \log(1+a))\ell T > 0$ for all ℓ sufficiently large (since $a - \log(1+a) > 0$), and so by Chebychev's inequality,

$$\limsup_{\ell\to\infty} \mathbb{Q}^\ell\left[|N_T^{\mathbb{Q}^\ell} - \ell T| \geq \frac{\log\frac{K}{S_0} - \sigma w + \frac{\sigma^2}{2}T + (a - \log(1+a))\ell T}{\log(1+a)}\right] \tag{33}$$

$$\leq \limsup_{\ell\to\infty} \frac{\ell T \log(1+a)^2}{(\log\frac{K}{S_0} - \sigma w + \frac{\sigma^2}{2}T + (a - \log(1+a))\ell T)^2} = 0. \tag{34}$$

This together with the above and dominated convergence establishes the claim.

(c) First, note that $S/S_0 = \mathcal{E}(R)$ is a true nonnegative martingale with mean 1 by Exercise 1-5 (a). So for $\ell > 0$ define $\widetilde{\mathbb{Q}}^\ell \approx \mathbb{Q}^\ell$ on $\mathscr{F}_T$ by $\frac{\mathrm{d}\widetilde{\mathbb{Q}}^\ell}{\mathrm{d}\mathbb{Q}^\ell} := S_T/S_0$. Note that

$$S_T/S_0 = \mathcal{E}(R)_T = \mathcal{E}(\sigma W^{\mathbb{Q}^\ell})_T \exp\left(\sum_{k=1}^{N_T^{\mathbb{Q}^\ell}} \log\frac{\widetilde{\ell}}{\ell} + (\ell - \widetilde{\ell})T\right), \tag{35}$$

where $\widetilde{\ell} := (1+a)\ell$. Now it follows as in part (a) that

$$R_t = \sigma W_t^{\mathbb{Q}^\ell} + aN_t^{\mathbb{Q}^\ell} - a\ell t = \sigma(W_t^{\widetilde{\mathbb{Q}}^\ell} + \sigma t) + aN_t^{\widetilde{\mathbb{Q}}^\ell} - a\ell t, \quad t \in [0,T], \tag{36}$$

where $W^{\widetilde{\mathbb{Q}}^\ell}$ is a $\widetilde{\mathbb{Q}}^\ell$-Brownian motion and $N^{\widetilde{\mathbb{Q}}^\ell} := N^{\mathbb{Q}^\ell}$ is a $\widetilde{\mathbb{Q}}^\ell$-Poisson process with rate $\widetilde{\ell} = (1+a)\ell$ and $W^{\widetilde{\mathbb{Q}}^\ell}$ and $N^{\widetilde{\mathbb{Q}}^\ell}$ are independent under $\widetilde{\mathbb{Q}}^\ell$.

So, for fixed $\ell > 0$, by Bayes' formula and independence of $W^{\widetilde{\mathbb{Q}}^\ell}$ and $N^{\widetilde{\mathbb{Q}}^\ell}$ under $\widetilde{\mathbb{Q}}^\ell$,

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}^\ell}[S_T \mathbb{1}_{\{S_T \le K\}}] = \mathbb{E}_{\widetilde{\mathbb{Q}}^\ell}[S_0 \mathbb{1}_{\{S_T \le K\}}] \\
&= S_0 \widetilde{\mathbb{Q}}^\ell\left[\exp\left(\sigma W_T^{\widetilde{\mathbb{Q}}^\ell} + \frac{\sigma^2}{2}T + \log(1+a)N_T^{\widetilde{\mathbb{Q}}^\ell} - a\ell T\right) \le \frac{K}{S_0}\right] \\
&= S_0 \widetilde{\mathbb{Q}}^\ell\left[N_T^{\widetilde{\mathbb{Q}}^\ell} - \widetilde{\ell}T \le \frac{\log\frac{K}{S_0} - \sigma W_T^{\widetilde{\mathbb{Q}}^\ell} - \frac{\sigma^2}{2}T - (\log(1+a)(1+a) - a)\ell T}{\log(1+a)}\right] \\
&\le S_0 \mathbb{Q}^\ell\left[\mathbb{Q}^\ell\left[|N_T^{\widetilde{\mathbb{Q}}^\ell} - \widetilde{\ell}T| \ge \frac{-\log\frac{K}{S_0} + \sigma w + \frac{\sigma^2}{2}T + (\log(1+a)(1+a) - a)\ell T}{\log(1+a)}\right]\Bigg|_{w = W_T^{\widetilde{\mathbb{Q}}^\ell}}\right] \\
&= S_0 \int_R \mathbb{Q}^\ell\left[|N_T^{\widetilde{\mathbb{Q}}^\ell} - \widetilde{\ell}T| \ge \frac{-\log\frac{K}{S_0} + \sigma w + \frac{\sigma^2}{2}T + (\log(1+a)(1+a)-a)\ell T}{\log(1+a)}\right] \frac{\exp(-\frac{w^2}{2T})}{\sqrt{2T\pi}}\,\mathrm{d}w.
\end{aligned} \tag{37}$$

Now, for fixed $w \in \mathbb{R}$, $-\log\frac{K}{S_0} + \sigma w + \frac{\sigma^2}{2}T + (\log(1+a)(1+a) - a)\ell T > 0$ for all ℓ sufficiently large (since $\log(1+a)(1+a) - a > 0$), and the claim follows by Chebychev's inequality and dominated convergence as in part (b).

(d) Since $S_0 + 1 \bullet S_T = S_T \ge (S_T - K)^+$ $\mathbb{P}$-a.s. and $K + 0 \bullet S_T = K \ge (K - S_T)^+$, it follows that $\Pi_s((S_T - K)^+) \le S_0$ and $\Pi_s((K - S_T)^+) \le K$. On the other hand, by Theorem 4.4 in the lecture notes and parts (b) and (c),

$$\begin{aligned}
\Pi_s((S_T - K)^+) &\ge \lim_{\ell\to\infty} \mathbb{E}_{\mathbb{Q}^\ell}[(S_T - K)^+] \\
&= \lim_{\ell\to\infty} \mathbb{E}_{\mathbb{Q}^\ell}[S_T \mathbb{1}_{\{S_T > K\}}] - K \lim_{\ell\to\infty} \mathbb{E}_{\mathbb{Q}^\ell}[\mathbb{1}_{\{S_T > K\}}] \\
&= S_0 - \lim_{\ell\to\infty} \mathbb{E}_{\mathbb{Q}^\ell}[S_T \mathbb{1}_{\{S_T \le K\}}] - 0 = S_0, \\
\Pi_s((K - S_T)^+) &\ge \lim_{\ell\to\infty} \mathbb{E}_{\mathbb{Q}^\ell}[(K - S_T)^+] \\
&= K \lim_{\ell\to\infty} \mathbb{E}_{\mathbb{Q}^\ell}[\mathbb{1}_{\{S_T \le K\}}] - \lim_{\ell\to\infty} \mathbb{E}_{\mathbb{Q}^\ell}[S_T \mathbb{1}_{\{S_T \le K\}}] \\
&= K - K \lim_{\ell\to\infty} \mathbb{E}_{\mathbb{Q}^\ell}[\mathbb{1}_{\{S_T > K\}}] - 0 = K.
\end{aligned} \tag{38}$$

Thus, $\Pi_s((S_T - K)^+) = S_0$ and $\Pi_s((K - S_T)^+) = K$, i.e., the superreplication strategy is the trivial buy-and-hold superhedge.