ETH Zürich HS 2013
D-MATH
Prof. J. Teichmann

Mathematical Finance

Exercise Sheet 4

Solution 4-1

For convenience, define the process $\widetilde{H} = (\widetilde{H}_k)_{k=0,\dots,T}$ by $\widetilde{H}_k := U(H_k)$, $k \in \{0,\dots,T\}$. Moreover, note that by a result in the lecture notes, $\overline{V}$ is a $\mathbb{P}$-supermartingale and satisfies

$$\overline{V}_k = \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_{k,T}} \mathbb{E}[U(H_\tau) \mid \mathscr{F}_k] \;\; \mathbb{P}\text{-a.s.} \tag{1}$$

(a) For $k = T$ the claim is trivial. So assume that $k < T$. Let $k \le \ell < T$. Using that $\overline{V}_\ell = \mathbb{E}[\overline{V}_{\ell+1} \mid \mathscr{F}_\ell]$ on $\{\tau_k^* > \ell\}$ since $\widetilde{H}_\ell < \overline{V}_\ell$ on $\{\tau_k^* > \ell\}$ and $\overline{V}^{\tau_k^*}_{\ell+1} = \overline{V}^{\tau_k^*}_\ell$ on $\{\tau_k^* \le \ell\}$ gives

$$\begin{aligned}\mathbb{E}[\overline{V}^{\tau_k^*}_{\ell+1} \mid \mathscr{F}_\ell] &= \mathbb{E}[\overline{V}_{\ell+1} \mid \mathscr{F}_\ell] \mathbb{1}_{\{\tau_k^* > \ell\}} + \overline{V}^{\tau_k^*}_\ell \mathbb{1}_{\{\tau_k^* \le \ell\}} \\ &= \overline{V}_\ell \mathbb{1}_{\{\tau_k^* > \ell\}} + \overline{V}^{\tau_k^*}_\ell \mathbb{1}_{\{\tau_k^* \le \ell\}} = \overline{V}^{\tau_k^*}_\ell \;\; \mathbb{P}\text{-a.s.}\end{aligned} \tag{2}$$

Thus, $\overline{V}^{\tau_k^*}$ is a $\mathbb{P}$-martingale on $\{k,\dots,T\}$. This together with the fact that $\overline{V}_{\tau_k^*} = \widetilde{H}_{\tau_k^*}$ by the definition of τ_k^* and (1) gives

$$\mathbb{E}[U(H_{\tau_k^*}) \mid \mathscr{F}_k] = \mathbb{E}[\overline{V}_{\tau_k^*} \mid \mathscr{F}_k] = \overline{V}^{\tau_k^*}_k = \overline{V}_k = \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_{k,T}} \mathbb{E}[U(H_\tau) \mid \mathscr{F}_k] \;\; \mathbb{P}\text{-a.s.} \tag{3}$$

(b) By the results in the lecture notes (adapted to our setup), there exist a predictable process $\vartheta = (\vartheta_k^1,\dots,\vartheta_k^d)_{k=1,\dots,T}$ and an increasing, adapted process $C = (C_k)_{k=0,\dots,T}$ null at 0 such that $\overline{V} = \overline{V}_0 + \vartheta \bullet S - C$ $\mathbb{P}$-a.s. Moreover, by part (a), the stopped process $\overline{V}^{\tau_0^*}$ is a $\mathbb{P}$-martingale. Thus,

$$\mathbb{E}[\vartheta \bullet S_{\tau_0^*} - C_{\tau_0^*}] = 0. \tag{4}$$

Since the local $\mathbb{P}$-martingale $\vartheta \bullet S$ is uniformly bounded from below by $-\overline{V}_0$, it is even a true $\mathbb{P}$-martingale by Exercise 3-1 (a). Hence, we may conclude that $\mathbb{E}[C_{\tau_0^*}] = 0$. Since C is nonnegative and increasing, this implies that $C \equiv 0$ $\mathbb{P}$-a.s. on $[\![0, \tau_0^*]\!]$. Then

$$\overline{V}_0 + \vartheta \bullet S_{\tau_0^*} = \overline{V}_{\tau_0^*} = \widetilde{H}_{\tau_0^*},$$

where the last equality follows from the definition of τ_0^*.

Solution 4-2

(a) First, suppose that there exists a buyer arbitrage. Then using that $\mathbb{Q} \approx \mathbb{P}$ on $\mathscr{F}_T$, there exist a predictable process $\vartheta = (\vartheta_k^1, \dots, \vartheta_k^d)_{k=1,\dots,T}$, a constant $c > 0$ and a stopping time $\tau \in \mathcal{S}_{0,T}$ such that

$$\vartheta \bullet S_\tau + c(H_\tau - S_0^H) \geq 0 \ \ \mathbb{Q}\text{-a.s.} \quad \text{and} \quad \mathbb{Q}[\vartheta \bullet S_\tau + c(H_\tau - S_0^H) > 0] > 0,$$

Thus,

$$\mathbb{E}_\mathbb{Q}[\vartheta \bullet S_\tau + c(H_\tau - S_0^H)] > 0. \tag{5}$$

Since H is bounded, it follows that $(\vartheta \bullet S_T^\tau)^- \in L^1(\mathbb{Q})$. Since $\vartheta \bullet S^\tau$ is a local $\mathbb{Q}$-martingale, Exercise 3-1 (a) implies that $\vartheta \bullet S^\tau$ is even a true $\mathbb{Q}$-martingale. Thus, $\mathbb{E}_\mathbb{Q}[\vartheta \bullet S_\tau] = 0$ and

$$\mathbb{E}_\mathbb{Q}[H_\tau] > S_0^H. \tag{6}$$

This shows that $S_0^H < \sup_{\tau \in \mathcal{S}_{0,T}} \mathbb{E}_\mathbb{Q}[H_\tau]$.

Conversely, suppose that $S_0^H < \sup_{\tau \in \mathcal{S}_{0,T}} \mathbb{E}_\mathbb{Q}[H_\tau]$. Let $\overline{V}$, τ_0^* and ϑ be as in Exercise 4-1 (b) (with $\mathbb{P}$ replaced by $\mathbb{Q}$ and $U(H)$ replaced by H). Then

$$(-\vartheta) \bullet S_{\tau_0^*} + (H_{\tau_0^*} - S_0^H) = \overline{V}_0 - S_0^H > 0 \ \ \mathbb{P}\text{-a.s.}, \tag{7}$$

and so there exists a buyer arbitrage.

(b) First, suppose that there exists a seller arbitrage. Then using that $\mathbb{Q} \approx \mathbb{P}$ on $\mathscr{F}_T$, there exist a predictable process $\vartheta = (\vartheta_k^1, \dots, \vartheta_k^d)_{k=1,\dots,T}$ and a constant $c < 0$ such that

$$\vartheta \bullet S_{\tau_0^*} + c(H_{\tau_0^*} - S_0^H) \geq 0 \ \ \mathbb{Q}\text{-a.s.} \quad \text{and} \quad \mathbb{Q}[\vartheta \bullet S_{\tau_0^*} + c(H_{\tau_0^*} - S_0^H) > 0] > 0 \tag{8}$$

where τ_0^* satisfies

$$\mathbb{E}_\mathbb{Q}[H_{\tau_0^*}] = \sup_{\tau \in \mathcal{S}_{0,T}} \mathbb{E}_\mathbb{Q}[H_\tau]. \tag{9}$$

The existence of τ_0^* follows from Exercise 4-1 (a). Then as in part (a), $\vartheta \bullet S^{\tau_0^*}$ is a true $\mathbb{Q}$-martingale and so

$$\mathbb{E}_\mathbb{Q}[H_{\tau_0^*}] < S_0^H. \tag{10}$$

Conversely, suppose that $S_0^H > \sup_{\tau \in \mathcal{S}_{0,T}} \mathbb{E}_\mathbb{Q}[H_\tau]$. By a result in the lecture notes, there exists a predictable process $\vartheta = (\vartheta_k^1, \dots, \vartheta_k^d)_{k=1,\dots,T}$ such that

$$\sup_{\tau \in \mathcal{S}_{0,T}} \mathbb{E}_\mathbb{Q}[H_\tau] + \vartheta \bullet S \geq H \ \ \mathbb{P}\text{-a.s.} \tag{11}$$

Thus, for each stopping time $\tau \in \mathcal{S}_{0,T}$,

$$\vartheta \bullet S_\tau - (H_\tau - S_0^H) \geq - \sup_{\tau \in \mathcal{S}_{0,T}} \mathbb{E}_\mathbb{Q}[H_\tau] + S_0^H > 0 \ \ \mathbb{P}\text{-a.s.}, \tag{12}$$

and so there exists a seller arbitrage.

Solution 4-3

(a) Since $r < \frac{\sigma^2}{2}$, $\widetilde{S}^1_t = s\exp\left(\sigma W_t + \left(r - \frac{\sigma^2}{2}\right)t\right)$ converges $\mathbb{P}$-a.s. to 0 as $t \to \infty$. Therefore $\tau_L < \infty$ $\mathbb{P}$-a.s. for all $L \in (0, K)$.

First, if $s \leq L$, then $\tau_L = 0$ and so $v_L(s) = K - s$. Next, if $s > L$, then $\tau_L = \sigma_{a,b}$ with $a = \frac{1}{\sigma}\log\frac{s}{L}$ and $b = \frac{\sigma}{2} - \frac{r}{\sigma}$. Moreover, $\widetilde{S}^1_{\tau_L} = L$ and therefore using the hint,

$$
\begin{aligned}
v_L(s) &= \mathbb{E}[\exp(-r\tau_L)(K - L)] \\
&= (K - L)\exp\left(-\frac{1}{\sigma}\log\frac{s}{L}\left(\sqrt{\frac{\sigma^2}{4} - r + \frac{r^2}{\sigma^2} + 2r} - \frac{\sigma}{2} + \frac{r}{\sigma}\right)\right) \\
&= (K - L)\left(\frac{s}{L}\right)^{-\frac{1}{\sigma}\left(\frac{\sigma}{2} + \frac{r}{\sigma} - \frac{\sigma}{2} + \frac{r}{\sigma}\right)} = (K - L)\left(\frac{s}{L}\right)^{-\frac{2r}{\sigma^2}}.
\end{aligned} \tag{13}
$$

Note that (13) also holds for $s = L$, i.e., the function $v_L(s)$ is continuous on $(0, \infty)$.

(b) First, define the function $g : (0, K) \to (0, \infty)$ by $g(L) := (K - L)L^{\frac{2r}{\sigma^2}}$. Then g is in $C^1((0, K))$ with $\lim_{L\downarrow\downarrow 0} g(L) = 0$ and $\lim_{L\uparrow\uparrow K} g(L) = 0$, and

$$
g'(L) = \frac{2r}{\sigma^2}KL^{\frac{2r}{\sigma^2}-1} - \left(\frac{2r}{\sigma^2} + 1\right)L^{\frac{2r}{\sigma^2}} = \frac{L^{\frac{2r}{\sigma^2}-1}}{\sigma^2}(2rK - (2r + \sigma^2)L). \tag{14}
$$

Solving for $g' = 0$ shows that $L^* := \frac{2r}{2r+\sigma^2}K$ is the unique maximiser of g in $(0, K)$. Second, for $L \in (0, K)$, define the function $h_L : (0, \infty) \to (0, \infty)$ by $h_L(s) = s^{-\frac{2r}{\sigma^2}}g(L)$. Then $h_{L^*}(L^*) = K - L^*$ and

$$
h'_{L^*}(L^*) = \frac{-2r}{\sigma^2 L^*}h_{L^*}(L^*) = \frac{-2r}{\sigma^2}\left(\frac{K - \frac{2r}{2r+\sigma^2}K}{\frac{2r}{2r+\sigma^2}K}\right) = -1. \tag{15}
$$

Third, for $L \in (0, K)$, note that $v_L(s) = (K - s)\mathbb{1}_{\{s\leq L\}} + h_L(s)\mathbb{1}_{\{s>L\}}$ for all $s \in (0, \infty)$. Since h_{L^*} is strictly convex, for $s \in (L^*, K]$,

$$
h_{L^*}(s) > h_{L^*}(L^*) + h'_{L^*}(L^*)(s - L^*) = (K - L^*) - (s - L^*) = K - s. \tag{16}
$$

This shows that $v_{L^*}(s) \geq (K - s)^+$ for all $s \in (0, \infty)$.

Finally, fix $L \in (0, K)$. We show that $v_L \leq v_{L^*}$. Indeed, by the above, for $s \leq L$,

$$
v_L(s) = K - s \leq v_{L^*}(s). \tag{17}
$$

Moreover, for $s \geq \max(L^*, L)$,

$$
v_L(s) = h_L(s) = s^{-\frac{2r}{\sigma^2}}g(L) \leq s^{-\frac{2r}{\sigma^2}}g(L^*) = h_{L^*}(s) = v_{L^*}(s). \tag{18}
$$

If $L \geq L^*$, this establishes the claim. Otherwise, let $s \in (L, L^*)$. Then there exists $\lambda \in (0, 1)$ such that $s = \lambda L + (1 - \lambda)L^*$. Then by convexity of h_L and using that $h_L \leq h_{L^*}$ as $g(L) \leq g(L^*)$,

$$
\begin{aligned}
v_L(s) = h_L(s) &= h_L(\lambda L + (1 - \lambda)L^*) \leq \lambda h_L(L) + (1 - \lambda)h_L(L^*) \\
&\leq \lambda h_L(L) + (1 - \lambda)h_{L^*}(L^*) = \lambda(K - L) + (1 - \lambda)(K - L^*) \\
&= K - s = v_{L^*}(s).
\end{aligned} \tag{19}
$$

(c) Define the function $f : (0,\infty) \to \mathbb{R}$ by $f(s) := K - s$. Then $v_{L^*} \in C^2((0,L^*) \cup (L^*,\infty))$ by the fact that $f, h_{L^*} \in C^2((0,\infty))$. Moreover, $v_{L^*} \in C^1((0,\infty))$ as $f(L^*) = h_{L^*}(L^*)$ and $f'(L^*) = h'_{L^*}(L^*)$. For $s \in (0,\infty)$, a simple differentiation gives

$$-rf(s) + rsf'(s) + \frac{1}{2}\sigma^2 s^2 f''(s) = -r(K-s) - rs + 0 = -rK \leq 0,$$
$$-rh_{L^*}(s) + rsh'_{L^*}(s) + \frac{1}{2}\sigma^2 s^2 h''_{L^*}(s) = g(L^*)s^{\frac{-2r}{\sigma^2}}\left(-r - \frac{2r^2}{\sigma^2} - r\left(\frac{-2r}{\sigma^2} - 1\right)\right) = 0. \quad (20)$$

This implies that

$$-rv_{L^*}(s) + rsv'_{L^*}(s) + \frac{1}{2}\sigma^2 s^2 v''_{L^*}(s) \leq 0, \quad s \in (0,\infty) \setminus \{L^*\}. \quad (21)$$

Next, by Itô's formula using the hint,

$$\begin{aligned} \mathrm{d}\widetilde{V}_t &= -r\exp(-rt)v_{L^*}(\widetilde{S}^1_t)\,\mathrm{d}t + \exp(-rt)v'_{L^*}(\widetilde{S}^1_t)\,\mathrm{d}\widetilde{S}^1_t \\ &\quad + \frac{1}{2}\exp(-rt)v''_{L^*}(\widetilde{S}^1_t)\mathbb{1}_{\{\widetilde{S}^1_t \neq L^*\}}\,\mathrm{d}\langle \widetilde{S}^1\rangle_t \\ &= \exp(-rt)\Big[\sigma v_{L^*}(\widetilde{S}^1_t)\widetilde{S}^1_t\,\mathrm{d}W_t \\ &\quad + \Big(-rv_{L^*}(\widetilde{S}^1_t) + rv'_{L^*}(\widetilde{S}^1_t)\widetilde{S}^1_t + \frac{1}{2}\sigma^2(\widetilde{S}^1_t)^2 v''_{L^*}(\widetilde{S}^1_t)\mathbb{1}_{\{\widetilde{S}^1_t \neq L^*\}}\Big)\mathrm{d}t\Big]. \end{aligned} \quad (22)$$

It follows from (21) that $\widetilde{V}$ is a local $\mathbb{P}$-supermartingale. Since it is nonnegative, Fatou's lemma implies that it is even a true $\mathbb{P}$-supermartingale.

Finally, by part (b), $\exp(-rt)v_{L^*}(s) \geq \exp(-rt)(K-s)^+$ for all $s \in (0,\infty)$ and $t \geq 0$. This together with the stopping theorem for supermartingales gives,

$$v_{L^*}(s) \geq \sup_{\tau \in \mathcal{S}_{0,\infty}} \mathbb{E}[\exp(-r\tau)v_{L^*}(\widetilde{S}^1_\tau)] \geq \sup_{\tau \in \mathcal{S}_{0,\infty}} \mathbb{E}\left[\frac{(K - \widetilde{S}^1_\tau)^+}{\widetilde{S}^0_\tau}\right] = v(s). \quad (23)$$

Since trivially $v_{L^*}(s) \leq v(s)$ for all $s \in (0,\infty)$, this establishes the claim.

Solution 4-4

(a) "$\Rightarrow$": Seeking a contradiction, suppose that S fails NA. Then there exists $\vartheta \in \mathbb{R}^d \setminus \{0\}$ such that $\vartheta^{tr}\Delta S_1 \geq 0$ $\mathbb{P}$-a.s. and $\mathbb{P}[\vartheta^{tr}\Delta S_1 > 0] > 0$. In particular, $\vartheta \in \mathcal{A}(0)$. But then also for each $\lambda > 0$, $\lambda\vartheta \in \mathcal{A}(0)$, and so $\mathcal{A}(0)$ is not bounded and hence not compact. Since $\mathcal{A}(0) \subset \mathcal{A}(x)$, we arrive at a contradiction.

"$\Leftarrow$": Seeking a contradiction, suppose that $\mathcal{A}(x)$ is not compact. Since $\mathcal{A}(x)$ is clearly closed, this means that $\mathcal{A}(x)$ is not bounded. Hence, there exists a sequence $(\vartheta_n)_{n\in\mathbb{N}}$ in $\mathcal{A}(x) \setminus \{0\}$ such that $\lim_{n\to\infty} \|\vartheta_n\|_\infty = +\infty$. For $n \in \mathbb{N}$, define $\eta_n := \frac{\vartheta_n}{\|\vartheta_n\|_\infty}$. Then $\|\eta_n\|_\infty = 1$ by construction for each $n \in \mathbb{N}$. Since the unit ball (with respect to the maximum norm) in $\mathbb{R}^d$ is compact, there exists a subsequence, denoted also by $(\eta_n)_{n\in\mathbb{N}}$, converging to some $\eta \in \mathbb{R}^d$ with $\|\eta\|_\infty = 1$. Using that $\vartheta_n \in \mathcal{A}(x)$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \|\vartheta_n\|_\infty = +\infty$ gives

$$\eta^{tr}\Delta S_1 = \lim_{n\to\infty} \eta_n^{tr}\Delta S_1 = \lim_{n\to\infty} \frac{\vartheta_n^{tr}\Delta S_1}{\|\vartheta_n\|_\infty} \geq \liminf_{n\to\infty} \frac{-x}{\|\vartheta_n\|_\infty} = 0 \ \ \mathbb{P}\text{-a.s.} \quad (24)$$

Since $\eta \neq 0$, it follows from the non-redundancy of S that $\mathbb{P}[\eta^{tr}\Delta S_1 > 0] > 0$. Thus, η is an arbitrage opportunity, and we arrive at a contradiction.

(b) "⇒": Seeking a contradiction, suppose that S fails NA. Then there exists $\vartheta \in \mathbb{R}^d \setminus \{0\}$ such that $\vartheta^{tr}\Delta S_1 \geq 0$ $\mathbb{P}$-a.s. and $\mathbb{P}[\vartheta^{tr}\Delta S_1 > 0] > 0$. Then by monotone convergence and by the fact that $U(\infty) = +\infty$,

$$\lim_{\lambda\to\infty} \mathbb{E}[U(x + \lambda\vartheta^{tr}\Delta S_1)] = U(x)\mathbb{P}[\vartheta^{tr}\Delta S_1 = 0] + U(\infty)\mathbb{P}[\vartheta^{tr}\Delta S_1 > 0] = +\infty, \tag{25}$$

Since $\lambda\vartheta \in \mathcal{A}(x)$ for all $\lambda > 0$ as in part (a), this implies that $u(x) = +\infty$, and we arrive at a contradiction.

"⇐": Since $\mathcal{A}(x)$ is compact by part (a), there exists $c > 0$ such that $\|\vartheta\|_\infty \leq c$ for all $\vartheta \in \mathcal{A}(x)$. This together with concavity of U shows that for all $\vartheta \in \mathcal{A}(x)$,

$$U(x + \vartheta^{tr}\Delta S_1) \leq U(x) + U'(x)(\vartheta^{tr}\Delta S_1) \leq U(x) + cU'(x)\sum_{i=1}^{d} |\Delta S_1^i| =: Y. \tag{26}$$

Note that Y is integrable since $\mathbb{E}[|\Delta S_1^i|] < \infty$ for $i \in \{1, \ldots, d\}$ by hypothesis and by the fact that $\mathscr{F}_0$ is trivial. Thus

$$u(x) = \sup_{\vartheta\in\mathcal{A}(x)} \mathbb{E}[U(x + \vartheta^{tr}\Delta S_1)] \leq \mathbb{E}[Y] < \infty. \tag{27}$$

(c) Note that $u(x) < \infty$ by part (b).

First, we establish existence of ϑ^*. Let $(\vartheta_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{A}(x)$ such that

$$\lim_{n\to\infty} \mathbb{E}[U(x + \vartheta_n^{tr}\Delta S_1)] = u(x). \tag{28}$$

Since $\mathcal{A}(x)$ is compact by part (a), there exists a subsequence, denoted again by $(\vartheta_n)_{n\in\mathbb{N}}$, converging to some $\vartheta^* \in \mathcal{A}(x)$. Now by Fatou's lemma using (26), and the fact that $\vartheta^* \in \mathcal{A}(x)$,

$$\begin{aligned} u(x) &= \lim_{n\to\infty} \mathbb{E}[U(x + \vartheta_n^{tr}\Delta S_1)] \leq \mathbb{E}\left[\limsup_{n\to\infty} U(x + \vartheta_n^{tr}\Delta S_1)\right] \\ &= \mathbb{E}\left[U(x + (\vartheta^*)^{tr}\Delta S_1)\right] \leq u(x). \end{aligned} \tag{29}$$

Next, we establish uniqueness of ϑ^*. To this end, let $\widetilde{\vartheta}^* \in \mathcal{A}(x)$ be another maximiser of $\mathbb{E}[U(x + \vartheta^{tr}\Delta S_1)]$. Set $\widehat{\vartheta}^* := \frac{1}{2}\vartheta^* + \frac{1}{2}\widetilde{\vartheta}^*$. Then $\widehat{\vartheta}^* \in \mathcal{A}(x)$ by convexity of $\mathcal{A}(x)$. By concavity of U on $[0, \infty)$,

$$U(x + (\widehat{\vartheta}^*)^{tr}\Delta S_1) \geq \frac{1}{2}U(x + (\vartheta^*)^{tr}\Delta S_1) + \frac{1}{2}U(x + (\widetilde{\vartheta}^*)^{tr}\Delta S_1). \tag{30}$$

Moreover, by strict concavity of U on $(0, \infty)$, by strict concavity of U on $[0, \infty)$ in case that $U(0) > -\infty$ and by the fact that $x + (\vartheta^*)^{tr}\Delta S_1, x + (\widetilde{\vartheta}^*)^{tr}\Delta S_1 > 0$ $\mathbb{P}$-a.s. in case that $U(0) = -\infty$, the inequality in (30) is strict on $\{(\vartheta^*)^{tr}\Delta S_1 \neq (\widetilde{\vartheta}^*)^{tr}\Delta S_1\}$. On the other hand, by maximality of ϑ^* and $\widetilde{\vartheta}^*$, it follows that

$$\mathbb{E}[U(x + (\widehat{\vartheta}^*)^{tr}\Delta S_1)] \leq \frac{1}{2}\mathbb{E}[U(x + (\vartheta^*)^{tr}\Delta S_1)] + \frac{1}{2}\mathbb{E}[U(x + (\widetilde{\vartheta}^*)^{tr}\Delta S_1)].$$

Thus, we may conclude that $(\vartheta^*)^{tr}\Delta S_1 = (\widetilde{\vartheta}^*)^{tr}\Delta S_1$ $\mathbb{P}$-a.s. Now non-redundancy of S gives $\widetilde{\vartheta}^* = \vartheta^*$.

Solution 4-5

(a) Fix $0 \le a < b < c$. Then there exists $\lambda \in (0,1)$ such that $b = \lambda c + (1-\lambda)a$. By concavity of U,

$$\begin{aligned}\frac{U(b)-U(a)}{b-a} &= \frac{U(\lambda c+(1-\lambda)a)-U(a)}{\lambda(c-a)} \ge \frac{\lambda(U(c)-U(a))}{\lambda(c-a)} = \frac{U(c)-U(a)}{c-a}\\ &= \frac{(1-\lambda)(U(c)-U(a))}{(1-\lambda)(c-a)} \ge \frac{U(c)-U(\lambda c+(1-\lambda)a)}{(1-\lambda)(c-a)} = \frac{U(c)-U(b)}{c-b}. \end{aligned} \tag{31}$$

For $z < y' < y''$, setting $a := z$, $b := y'$ and $c := y''$ shows that $y \mapsto \frac{U(y)-U(z)}{y-z}$ is decreasing on (z,∞), for $y' < y'' < z$, setting $a := y'$, $b := y''$ and $c := z$ shows that $y \mapsto \frac{U(y)-U(z)}{y-z}$ is also decreasing on $(0,z)$, and for $y' < z < y''$, setting $a := y'$, $b := z$ and $c := y''$, establishes that $y \mapsto \frac{U(y)-U(z)}{y-z}$ is decreasing everywhere on $(0,\infty)\setminus\{z\}$.

(b) Let $\eta \in \mathbb{R}^d \setminus \{0\}$ be arbitrary. Since ϑ^* is an interior point of $\mathcal{A}(x)$, $\vartheta^* + \epsilon\eta \in \mathcal{A}(x)$ for all $\epsilon > 0$ sufficiently small. For $\epsilon > 0$ sufficiently small, set

$$\Delta_\epsilon^\eta := \frac{U(x+(\vartheta^*+\epsilon\eta)^{tr}\Delta S_1) - U(x+(\vartheta^*)^{tr}\Delta S_1)}{\epsilon}. \tag{32}$$

Then on $\{\eta^{tr}\Delta S_1 \neq 0\}$,

$$\Delta_\epsilon^\eta = (\eta^{tr}\Delta S_1)\frac{U(x+(\vartheta^*+\epsilon\eta)^{tr}\Delta S_1) - U(x+(\vartheta^*)^{tr}\Delta S_1)}{\epsilon\eta^{tr}\Delta S_1}, \tag{33}$$

and by part (a), this increases monotonically to $(\eta^{tr}\Delta S_1)U'(x+(\vartheta^*)^{tr}\Delta S_1) > -\infty$ as $\epsilon \downarrow 0$. In particular, for $\eta := \vartheta^*$, using that $U' < +\infty$ on $(0,\infty)$ and $(\vartheta^*)^{tr}\Delta S_1 = -x < 0$ on $\{x+(\vartheta^*)^{tr}\Delta S_1 = 0\}$, this gives $U'(x+(\vartheta^*)^{tr}\Delta S_1) < \infty$ $\mathbb{P}$-a.s.

On the other hand, on $\{\eta^{tr}\Delta S_1 = 0\}$, $\Delta_\epsilon^\eta \equiv 0$, and this trivially increases monotonically to $(\eta^{tr}\Delta S_1)U'(x+(\vartheta^*)^{tr}\Delta S_1)$ as $\epsilon \downarrow 0$.

Now by the fact that U is increasing, by the fact that $U(0) > -\infty$ and by optimality of ϑ^*, for $\epsilon > 0$ sufficiently small,

$$\frac{U(0) - U(x+(\vartheta^*)^{tr}\Delta S_1)}{\epsilon} \le \Delta_\epsilon^\eta. \tag{34}$$

Thus, $\Delta_\epsilon^\eta \in L^1(\mathbb{P})$ for ϵ sufficiently small, and so by the above and monotone convergence, $(\eta^{tr}\Delta S_1)U'(x+(\vartheta^*)^{tr}\Delta S_1) \in L^1(\mathbb{P})$ and

$$\mathbb{E}[(\eta^{tr}\Delta S_1)U'(x+(\vartheta^*)^{tr}\Delta S_1)] \le 0. \tag{35}$$

The final claim follows by setting $\eta := (1,0,\dots,0)$, $\eta = (-1,0,\dots,0)$, $\eta := (0,1,0,\dots,0)$, $\eta := (0,-1,0,\dots,0)$, $\dots$, $\eta := (0,\dots,0,1)$ and $\eta := (0,\dots,0,-1)$.

(c) Using that $U'(x+(\vartheta^*)^{tr}\Delta S_1) \in (0,\infty)$ $\mathbb{P}$-a.s. by strict concavity of U on $(0,\infty)$ and part (b) and that $\mathbb{E}[U'(x+(\vartheta^*)^{tr}\Delta S_1)\Delta S_1^i] = 0$ for all $i \in \{1,\dots,d\}$, it suffices to show that $U'(x+(\vartheta^*)^{tr}\Delta S_1) \in L^1(\mathbb{P})$. Since U' is decreasing on $(0,\infty)$, it even suffices to show that

$$U'(x+(\vartheta^*)^{tr}\Delta S_1)\mathbb{1}_{\{x+(\vartheta^*)^{tr}\Delta S_1 \le x/2\}} \in L^1(\mathbb{P}). \tag{36}$$

Since $((\vartheta^*)^{tr}\Delta S_1)U'(x+(\vartheta^*)^{tr}\Delta S_1) \in L^1(\mathbb{P})$ by part (b),

$$\begin{aligned}&\mathbb{E}[U'(x+(\vartheta^*)^{tr}\Delta S_1)\mathbb{1}_{\{x+(\vartheta^*)^{tr}\Delta S_1 \le x/2\}}]\\ &\qquad = \mathbb{E}[U'(x+(\vartheta^*)^{tr}\Delta S_1)\mathbb{1}_{\{(\vartheta^*)^{tr}\Delta S_1 \le -x/2\}}]\\ &\qquad \le \frac{\mathbb{E}[-((\vartheta^*)^{tr}\Delta S_1)U'(x+(\vartheta^*)^{tr}\Delta S_1)\mathbb{1}_{\{(\vartheta^*)^{tr}\Delta S_1 \le -x/2\}}]}{x/2}\\ &\qquad \le \frac{2}{x}\mathbb{E}[|(\vartheta^*)^{tr}\Delta S_1|U'(x+(\vartheta^*)^{tr}\Delta S_1)] < \infty. \end{aligned} \tag{37}$$