Exercise 0

1. Let $D := (a,b)$, $a < b \in \mathbb{R}$, and let $f, \alpha \in C^0(\bar{D})$. We consider the boundary-value problem

$$\begin{aligned} -u''(x) + \alpha(x)u(x) &= f(x) \\ u'(a) &= u_0 \\ u(b) &= u_1, \end{aligned} \tag{1}$$

where $u_0, u_1 \in \mathbb{R}$ are given constants.

a) Similarly to the lecture, deduce a *variational formulation* for (1). Proceed the following way:

1. Multiply the first equation by an arbitrary function $v \in C_0^\infty(D)$.
2. Use integration by parts until you obtain an integral term of the form $\int_a^b u'(x)v'(x)\,dx$.
3. Plug in the boundary conditions to obtain a variational formulation.

b) Instead of solving the differential equation (1), we look for solutions u to the variational equation derived in the previous subproblem.
In which space do solutions to this variational equation lie? Or, asked in a different manner, what conditions do solutions u to the variational equation need to satisfy? Answer the same question for the original equation (1) and compare.

c) Now we want to discretize the variational problem on the mesh $\mathcal{T}$ which is defined by $a =: x_0 < x_1 < \cdots < x_M := b$. As done in the lecture, consider $S^1(D, \mathcal{T})$, the finite-dimensional space of piecewise polynomials of degree 1, and state the corresponding discrete formulation in this space.

d) Choose the *hat-basis* $\{b_j(x) \;:\; j = 0, \ldots, M\}$ of the space $S^1(D, \mathcal{T})$ as introduced in the lecture. The discrete problem which you derived in the previous subproblem can be formulated as a matrix equation

$$\mathbf{A}\mathbf{u} = \mathbf{l},$$

where $\mathbf{u}$ is the $(M+1) \times 1$ vector of basis coefficients of the discrete solution $u_M(x)$ with respect to the hat basis.
Give precise definitions of the entries of the matrix $\mathbf{A}$ and the vector $\mathbf{l}$.

Please turn sheet!

2. For $u \in C^\infty([a,b])$ with $u(a) = 0$, show

$$\|u\| \leq \frac{b-a}{\sqrt{2}} \|u'\| \,,$$

where $\|f\| := \sqrt{\int_a^b |f(x)|^2 \, dx}$.

Hint: Consider $f(x) = \int_a^x f'(y) \, dy$, and use the Cauchy-Schwarz inequality to prove that $|f(x)|^2 \leq (x-a)\|f'\|^2$.

3. Let $D = (a,b)$. In the spirit of part b) of the first exercise, we define

$$H^1(D) := \Big\{ v \in L^2(D) \;:\; \exists w \in L^2(D), s.t. \int_a^b w(x)\varphi(x)\,dx = -\int_a^b v(x)\varphi'(x)\,dx \quad \forall \varphi \in C^\infty(\bar{D}) \text{ with } \varphi(a) = \varphi(b) = 0 \Big\} \;.$$

Note that the equation in the definition of $H^1(D)$ looks like the rule of integration by parts. Therefore, for a function $v \in H^1(D)$, the corresponding w as in the definition above will be called *(weak) derivative* of v.

a) Is it true that $f \in C^1([a,b])$ always implies $f \in H^1(a,b)$?

b) Let $D := (-1,1)$. Does the function $u(x) = |x|$ lie in $H^1(-1,1)$? If yes, what is its weak derivative?

c) Does $u(x) = \text{sign}(x)$ lie in $H^1(-1,1)$?

d) Let $D := (0,1)$.
For which exponents $\alpha > 0$ does the function $u(x) := x^\alpha$ lie in $L^2(D)$? For which exponents $\alpha > 0$ does the (usual) derivative $u'(x) = \alpha x^{\alpha-1}$ lie in $L^2(D)$? With the help of subproblem a), for which α is $u \in H^1(D)$?

Please justify your answers and give precise references to your sources (lecture notes, text books, exercise class, etc.). Consult the lecture homepage (`www.math.ethz.ch`) for administrative and further questions.

Due date: **Mon**, **Sept 23**, 2013