Exercise 1

1. Let $D := (a,b)$, $a < b \in \mathbb{R}$, and let $f, \alpha \in C^0(\bar{D})$. We consider the variational problem

Find $u \in V := \{w \in H^1(D) \;:\; w(b) = 0\}$, such that

$$\int_a^b u'(x)v'(x)\,dx + \int_a^b \alpha(x)u(x)v(x)\,dx = \int_a^b f(x)v(x)\,dx \tag{1}$$

a) Prove that if $f, \alpha \in C^0(\bar{D})$ and $u \in C^2(\bar{D})$,

$$-u''(x) + \alpha(x)u(x) = f(x) \;\; \forall x \in D, \text{ and } u'(a) = 0, u(b) = 0\ .$$

b) Let $K_i := (x_{i-1}, x_i)$. Find an affine map $F_{K_i} : [-1,1] \to \bar{K}_i$ which is bijective and maps -1 to x_{i-1} and 1 to x_i.

c) (In what follows, we adopt the notation of Sheet 0 and the lecture.)
By Sheet 0, Ex. 1d, the discretization with piecewise linear FEM and the hat basis yields the linear system $\mathbf{A}\mathbf{u} = \mathbf{l}$, being $\mathbf{u}_h$ the vector of basis coefficients of u_h, and

$$(\mathbf{A})_{ij} := \underbrace{\int_a^b b_j'(x)b_i'(x)\,dx}_{=:(\mathbf{K})_{ij}} + \underbrace{\int_a^b \alpha(x)b_i(x)b_j(x)\,dx}_{=:(\mathbf{M})_{ij}}, \quad \text{and } (\mathbf{l})_i := \int_a^b f(x)b_i(x)\,dx\ .$$

Now, assume that no Dirichlet boundary conditions are given. Prove that

$$(\mathbf{K})_{ij} = \begin{cases} 0 & |i-j| \geq 2 \\ -\frac{1}{h} & |i-j| = 1 \\ \frac{2}{h} & i = j, i \notin \{0, M\} \\ \frac{1}{h} & i = j, i \in \{0, M\}\ . \end{cases}$$

Deduce a special structure for $\mathbf{K}$.
We call $\mathbf{K}$ the *stiffness matrix* and for $\alpha(x) = 1$, we call $\mathbf{M}$ the *mass matrix*.

d) We define the *shape functions* $\hat{N}_0(\xi) := \frac{1-\xi}{2}$, $\hat{N}_1(\xi) := \frac{1+\xi}{2}$. In order to compute $\mathbf{K}$, we propose the following algorithm (called *assembly*):

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for $i = 1, \dots, M$

1. Compute $(\mathbf{K}_{K_i})_{k,l} := \frac{2}{|x_i - x_{i-1}|} \int_{-1}^{1} \hat{N}_k'(x)\hat{N}_l'(x)\,dx$, for $k, l = 1, 2$.
2. Update the values to the global matrices. Let $glob(k) := \begin{cases} i-1 & k=1 \\ i & k=2 \end{cases}.$

$$(\mathbf{K})_{glob(k),glob(l)} := (\mathbf{K})_{glob(k),glob(l)} + (\mathbf{K}_{K_i})_{k,l}$$

Prove that this algorithm yields the same $\mathbf{K}$ as computed in subproblem c).

e) In order to compute $\mathbf{K}_{K_i}$ efficiently, one usually computes the integrals on $[-1, 1]$ by hand first. Compute $\int_{-1}^{1} \hat{N}_k'(x)\hat{N}_l'(x)\,dx$ and $\int_{-1}^{1} \hat{N}_k(x)\hat{N}_l(x)\,dx$, $k = 1, 2$.

f) The Matlab function `assemMat_STIMA_LFE1D.m` (website) computes $\mathbf{K}$ by assembly. Write a function `assemMat_MASS_LFE1D.m` that computes $\mathbf{M}_{K_i}$ *for constant* $\alpha(x) = \alpha$. There is a skeleton on the website.

g) Write a function `assemLoad_LFE1D.m` (skeleton on website) that assembles the *load vector* $\mathbf{l}$ by assembly. For general f, we need a quadrature rule.

h) Finally! Write a Matlab-code `Call_Poiss_LFE1D.m` (skeleton on the page) that solves problem (1) by linear FEM on $D := (-1/2, 1)$, with $u(x) = \sin(\omega\pi x)$, $\omega = 3$, and with $\alpha \equiv 1$, on the grid $x_i := \frac{i}{M}$, $i = 0, \dots, M$.
Do the computations for $M = 2^n$, $n = 5, 10, 100, 250, 500, 750, 1000$. For each n, compute the error norm $\|u - u_h\|_{H^1(D)}$ with the function `L2Err_LFE1D.m` on the website. Repeat for $\omega = 5, 7, \dots$. What happens to the convergence?

2. Let $D := (a, b)$, $u \in H^2(D)$ and let $u_I \in S^1(D, \mathcal{T})$ be the piecewise linear interpolant of u at the points $\mathcal{T} := \{a = x_0 < x_1 < \cdots < x_M = b\}$. It is well-defined by $u_I(x_j) = u(x_j)$, for $0 \le j \le M$.

a) Prove that for all $i = 1, \dots, M$,

$$|u - u_I|_{H^1((x_{i-1},x_i))} \le c\,|x_i - x_{i-1}|\,\|u\|_{H^2((x_{i-1},x_i))}.$$

Hint: Let $e := u - u_I$. The problem can be reduced to $\int_0^1 \tilde{e}'(\xi)^2\,d\xi \le c \int_0^1 \tilde{e}''(\xi)^2\,d\xi$. A "standard trick" you can use is $\left|\int_x^y w(z)\,dz\right| \le c' \int_x^y |w(z)|^2\,dz$ (prove!).

b) Deduce from a) that $\|u - u_I\|_{H^1(D)} \le c'' \max_{i=1,\dots,M} |x_i - x_{i-1}|\,\|u\|_{H^2(D)}$.

Please justify your answers and give precise references to your sources (lecture notes, text books, exercise class, etc.). Consult the lecture homepage (`www.math.ethz.ch`) for administrative and further questions.

Due date: **Mon**, **Sept 23**, 2013